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Three Important Theorems for Analytic Functions

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THESIS

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THREE IMPORTANT THEOREMS FOR ANALYTIC FUNCTIONS

THESIS

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Texas Southern University, 2000

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This study lists the basic definitions and concepts relating to analytic functions, indicates the fundamental theorems which are assumed without proof, and then develops and proves the necessary preliminary results needed for the following

Theorems:

1. MAXIMUM MODULUS THEOREM
2. OPEN MAPPING THEOREM
3. INVERSE FUNCTION THEOREM

These three theorems form an important basis for the theoretical study of Julia sets and the Mandelbort set.

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CHAPTER 1

INTRODUCTION

One of the most fascinating and complex objects in mathematics is the Mandelbort set. This set contains rough copies of itself under magnification, but upon closer inspection none of the copies are exactly like the original set nor each other. A cataloguing of the different images within the Mandelbort set or a numerical description of its outline would require an infinity of information. Paradoxically, the set is generated by a very simple mathematical procedure – it takes a relatively short computer program to reproduce the entire set (1).

Let **C** represent the complex plane and f: $\mathbf{C} \to \mathbf{C}$ be a function. Any complex number b determines a complex sequence $\{z_n\}$, called the orbit of b generated by f, as follows: $z_0 = b,\ z_1 = f(z_0),\ z_2 = f(z_1), \ldots,\ z_n = f(z_{n-1}), \ldots$.

Julia sets arose in the study of the dynamics of complex quadratic functions $f(z) = z^2 + c$, where c is a complex number. The set of complex numbers whose orbits are bounded is called the filled Julia set B of the function f and the boundary of B is called the Julia set of the function (2). The Mandelbort set is the set of all points c in the complex plane such that the orbit of 0 generated by f is bounded (3).

The study of Julia sets and the Mandelbort set require the application of several important but rather deep results in analytic function theory (4). These results will be proved in this paper, after the development of the necessary supporting material relating to analytic functions (5, 6).

Chapter 2 provides the basic definitions and concepts relating to analytic functions along with some examples and theorems concerning these functions. The results in this chapter constitute the basis for this study and include the fundamental theorems, which are assumed without proof.

Chapter 3 establishes and proves the necessary preliminary results needed for the main theorems in Chapter 4. These include Cauchy's Integral Theorem and Liouville's Theorem. Cauchy's Integral Theorem is also presented without proof and may be found in most standard complex variable texts (5, 6, 7).

Chapter 4 presents rigorous proofs of the following three theorems:

1. Maximum Modulus Theorem: If f is analytic at z_0 such that f is not constant on any r-neighborhood of z_0, then for any positive number $\varepsilon > 0$, there exists a z in $N_\varepsilon(z_0)$ such that $|f(z)| > |f(z_0)|$. Thus z_0 cannot be a local maximum point of $|f(z)|$.

2. Open Mapping Theorem: Let f be analytic on the open connected set D such that $f'(z) \neq 0$ on D. Then for every open set U contained in D, the image of U, f(U), is an open set. Moreover, the image of D, is an open connected set.

3. Inverse Function Theorem: If f is analytic and one-to-one on the open connected D such that $f'(z) \neq 0$ on D, then the inverse function of f is an analytic function

on f(D). Moreover, if $f^{-1} = g$ denotes the inverse function, then

$g'(w) = \dfrac{1}{f'(g(w))}$ for w in f(D). Schwarz's Lemma is also presented in Chapter 4. Finally, Chapter 5 is a summary of this paper and contains a recommendation for a future study of Julia sets and the Mandelbort set.

CHAPTER 2

PRELIMINARY RESULTS

Some algebraic problems cannot be solved using only the real number system. For instance, there is no real number which is a solution to the quadratic equation $x^2 + 1 = 0$. To solve this and other similar problems, a new number system is defined, the complex numbers. A complex number is written in the form $A= a + bi$ where $i = \sqrt{-1}$. The real part of A, denoted by Re(A), is a; while the imaginary part, Im(A), is b.

Let **C** denote the set of complex numbers, $\mathbf{C} = \{a+bi | a,b \text{ are real numbers}\}$. The set of real numbers **R** is considered to be a subset of **C**, where

$\mathbf{R} = \{a+bi | a \text{ is real and } b=0\}$.

The complex conjugate of A is $\overline{A} = a - bi$ and the modulus or absolute value of A is $|A| = \sqrt{a^2+b^2}$. If $A = a+0i = a$ is a real number, then $|A| = |a|$ is the absolute value of a. If A and B are complex numbers, then $|A - B|$ is called the distance between A and B.

Properties of modulus $|A|$;

1- $|A|$ is a non negative real number,

2- $|A| = 0$ if and only if $A = 0$,

3- $|\overline{A}| = |-A| = |A|$,

4- $A\overline{A} = |A|^2$,

5- $|AB| = |A||B|$,

6- $\left|\frac{A}{B}\right| = \frac{|A|}{|B|}$, if $B \neq 0$,

7- $|A+B| \leq |A| + |B|$.

Let D be a subset of **C** and use z to denote a complex variable. If for every value z in D, f associates a unique complex number to z, denoted f(z), then f is called a complex valued function of a complex variable with domain D. Let $f(D) = \{f(z) | z \in D\}$ denote the image of D or the range of f.

A complex valued function of a complex variable z = x + iy is a combination of two real valued functions in x and y, $f(z) = u(x,y) + iv(x,y)$. For example if $f(z) = z^2$, then $f(z) = (x+iy)^2 = x^2 - y^2 + i2xy$ where $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. If w is a complex number, let $f^{-1}(w)$ denoted the set $f^{-1}(w) = \{z | f(z) = w\}$. A function f is said to be one-to-one provided that $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$. This is equivalent to the statement that $f^{-1}(w)$ consists of exactly one number for each w $\in$ f(D).

Some of the basic topological ideas concerning **C** are listed below. If $z_0 \in$ C and r > 0 is a positive real number, then the r - neighborhood of z_0 is the set $N_r(z_0) = \left\{ y \in C \,\middle|\, |y - z_0| < r \right\}$. $N_r(z_0)$ denotes the set of complex numbers whose distance from z_0 is less than r. If A is a set in **C**, then A is an open set if and only if for each $z_0 \in$ A there exists r > 0 such that $N_r(z_0) \subset A$. Moreover, $C_r(z_0) = \left\{ y \,\middle|\, |y - z_0| = r \right\}$ is the circle of radius r and center z_0 and $B_r(z_0) = \left\{ y \in C \,\middle|\, |z_0 - y| \le r \right\}$ is called a closed disc about z_0 .

A is called a closed set in **C** if and only if **C**-A is an open set, where

C - A = $\left\{ z \in C \,\middle|\, z \notin A \right\}$.

For r > 0, $N_r(z_0)$ is an open set and $B_r(z_0)$ is a closed set. Let A $\subset$ **C** then A is said to be bounded if and only if there exists r > 0 such that A $\subset$ $N_r(0)$. A compact set is usually defined in terms of open covers, but in **C** this concept is characterized in the following theorem.

<u>Heine-Borel Theorem 2.1</u>

A set A $\subset$ **C** is compact if and only if A is both a closed and bounded set. For example $B_r(z_0)$ is compact since this set is both closed and bounded.

The concept of limit is one of the most important ideas in analysis. The definitions of limit, continuity and the derivative for complex valued functions are essentially the same as for real valued functions.

Let D be an open set in **C** and f: D $\to$ **C** be a function. For $z_0 \in$ D and $w_0 \in$ **C**, then define the limit of f(z) as z approaches z_0 is w_0 , written $\lim_{z \to z_0} f(z) = w_0$, if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \text{implies} \quad |f(z) - f(z_0)| < \varepsilon .$$

Then f is said to be continuous at z_0 if and only if $\lim_{z \to z_0} f(z) = f(z_0)$. If A $\subset$ D is an open set, then f is continuous on A provided f is continuous at every point in A; moreover, if f is said to be continuous on its domain D, then f is said to be a continuous function.

If f: $D_1 \to$ **C**, g: $D_2 \to$ C and $z_0 \in D_1 \cap D_2$ such that f and g are continuous at z_0 , then f+g, f-g, and f.g are continuous at z_0. Furthermore, if $g(z_0) \neq 0$, then $\frac{f}{g}$ is also continuous at z_0. Polynomial functions $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ and rational functions $r(z) = \dfrac{P(z)}{Q(z)}$, where P and Q are polynomials, are continuous.

If f = u + iv where u, v are real valued functions, and if $w_0 = a_0 + ib_0$, then $\lim_{z \to z_0} f(z) = w_0$ if and only if $\lim_{z \to z_0} u(z) = a_0$ and $\lim_{z \to z_0} v(z) = b_0$. Thus, f is continuous at z_0 if and only if both u and v are continuous real valued functions at z_0.

A function f is bounded on A $\subset$ D if and only if f(A) is bounded set in **C**. Thus, there exists r > 0 such that $|f(z)| < r$ for all z $\in$ A.

Some important properties of continuous functions and compact sets are listed in the following theorem.

Theorem 2.2

If A $\subset$ D and if f is continuous and A is compact, then the set $f(A) = \{f(z) \mid z \in A\}$ (called the image of A) is a compact subset of **C**. Moreover, f is bounded on A and there exists u $\in$ A and v $\in$ A such that $|f(u)| \leq |f(z)| \leq |f(v)|$ for all z $\in$ A.

If $[a,b]$ is a closed bounded interval in **R,** then a continuous function $\gamma:[a,b] \to C$ is called a curve in **C** from γ(a) to γ(b). A curve is said to be closed if and only if γ(a) = γ(b). A curve is a simple closed curve if and only if γ is closed and γ is one-to-one on [a,b). Thus γ(a) = γ(b) and if $a \leq t_1 \leq t_2 < b$, then $\gamma(t_1) \neq \gamma(t_2)$. Hence γ will not cross itself but γ does end where it starts.

Let $\gamma:[a,b] \to C$ be a curve in **C**, then $\gamma([a,b]) = \{\gamma(t) | a \leq t \leq b\}$ is called the trace of γ and is denoted Tr(γ). The condition γ(t) $\in$ D for t $\in$ [a,b] is equivalent to

Tr(γ) $\subset$ D.

Example

If $z_0 \in$ **C** and r > 0, then define γ:[0,2π] → **C** by $\gamma(t) = z_0 + re^{it}$. The trace of γ is a circle centered at z_0 of radius r.

If γ: [a,b] → **C** and ρ : [c,d] → **C** are curves in **C** such that γ(b) = ρ(c), then a new curve, denoted γ # ρ, can be defined going along γ from γ(a) to γ(b) = ρ(c) and then along ρ to ρ(d).

A set D is said to be path connected if and only if for every two points in D there exists a curve in D from one to the other. In other words if z_0 , and z_1 are in D, then there exists a curve $\gamma:[a,b]\to C$ such that $\gamma(a)=z_0$, $\gamma(b) = z_1$ and $\mathrm{Tr}(\gamma)\subset D$. An open path connected set is called a domain.

Let $\gamma:[a,b]\to C$ be a curve. If a = $t_0<t_1<\ldots<t_n=b$, then $\{t_0,t_1,\ldots,t_n\}$ is called a partition on the interval [a,b]. The corresponding set $\{\gamma(t_0),\gamma(t_1),\ldots,\gamma(t_n)\}$ is a set of points located on the trace of γ and $\sum_{i=1}^{n}|\gamma(t_i)-\gamma(t_{i-1})|$ is the length of the polygonal arc joining these points. If the set of all such sums $\left\{\sum_{i=1}^{n}|\gamma(t_i)-\gamma(t_{i-1})| \mid \{t_0,t_1,\ldots,t_n\} \text{ is a partition of } [a,b]\right\}$ has an upper bound, then γ is said to be a rectifiable curve and the least upper of these sums is defined to be the length of γ, denoted L(γ). Thus, a curve is rectifiable if and only if it has finite length. If $\gamma(t)=\gamma_1(t)+i\gamma_2(t)$, then γ_1 and γ_2 are real valued functions of a real variable. Moreover, γ is continuous if and only if both γ_1 and γ_2 are continuous.

The derivative of γ is defined

$$\gamma'(t)=\gamma_1'(t)+i\gamma_2'(t) \quad \text{where} \quad \gamma_i'(t)=\frac{d\gamma_i}{dt}.$$

If γ' exists and is continuous, then γ is called a smooth curve. If γ is a smooth curve, then γ is rectifiable and the length of γ equals $L(\gamma)=\int_a^b|\gamma'(t)|dt$.

Let $f: D \to C$ be a complex valued function of a complex variable and $\gamma:[a,b] \to D$ be a curve. If f is continuous and γ is rectifiable, then the integral of f along γ , denoted $\int_\gamma f(z)dz$ or $\int_\gamma f$, exists and is a complex number.

Basically, $\int_\gamma f$ is the limit of sums of the form $\sum_{i=1}^{n} f(z_i)(\gamma(t_i) - \gamma(t_{i-1}))$ where $\{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$ and $z_i = \gamma(s_i)$ for $t_{i-1} \le s_i \le t_i$. The limit is taken as the max $\{|t_i - t_{i-1}| \mid i = 1,2,\ldots,n\}$ goes to 0. More precisely, if f is continuous and γ is rectifiable, then there exists a complex number I satisfying; for ε > 0 there exists δ > 0 such that if $\{t_0, t_1, \ldots, t_n\}$ is any partition of [a,b] with $0 < t_i - t_{i-1} < \delta$ for i = 1,2,…,n and if $s_i \in [t_i, t_{i-1}]$, then

$$\left|\sum_{i=1}^{n} f(\gamma(s_i))(\gamma(t_i) - \gamma(t_{i-1})) - I\right| < \varepsilon.$$

The number I is called the integral of f along γ and is denoted $\int_\gamma f(z)dz$.

Properties of the Integral

If f and g are continuous, γ is rectifiable and a,b are constants, then

(1) $\int_\gamma \left(af(z) + bg(z)\right)dz = a\int_\gamma f(z)dz + b\int_\gamma g(z)dz$.

(2) If $\gamma = \gamma_1 \# \gamma_2 \# \gamma_3 \# \ldots \# \gamma_n$, then $\int_\gamma f(z)dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz$.

(# means breaking the integral along γ up into the sum of integrals taken along the pieces $\gamma_1, \gamma_2, \ldots, \gamma_n$ of γ)

(3) If $|f(z)| \le M$ for every $z \in \mathrm{Tr}(\gamma)$, then $\left|\int_\gamma f(z)dz\right| \le ML(\gamma)$

Let f be a function with domain D, an open subset of **C**, and let $z_0 \in$ D. Then f is said to be differentiable at z_0 if and only if the limit,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

This limit is called the derivative of f at z_0 and is denoted $f'(z_0)$. If f is differentiable at each point z in an open set A, then f is said to be differentiable on A. If f is differentiable on its domain, then f is said to be differentiable.

The Fundamental Theorem of Calculus 2.3

Let D be a domain in **C**, γ:[a,b] $\to$ D be a rectifiable curve in D and f be a complex valued function continuous on D . If F is a function such that $F'(z) = f(z)$ for $z \in$ D , then $\int_\gamma f dz = F(\gamma(b)) - F(\gamma(a))$.

Thus, if γ is a closed rectifiable curve and F' is continuous in a neighborhood of γ, then $\int_\gamma F'dz = 0$. This is true since $\gamma(a) = \gamma(b)$ implies $F(\gamma(b)) - F(\gamma(a)) = 0$.

A function f is said to be analytic at z if and only if there exists r > 0 such that f is differentiable on $N_r(z_0)$. Also f is analytic on A if and only if it is analytic at each point in A, and it is analytic if it is analytic on its domain.

What is the difference between f being differentiable at z_0 and f being analytic at z_0? If f is analytic at z_0, then f is not only differentiable at z_0 but is also differentiable at all point near z_0, which means at all points in a neighborhood of z_0. Note if f is analytic at z_0, then f is analytic on some neighborhood of z_0.

The usual differentiation formulas for real variables are also valid for complex variables. For example, if

$P(z) = \sum_{k=0}^{n} a_k z^k = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ is a complex polynomial, then

$$P'(z) = \sum_{k=1}^{n} k a_k z^{k-1} = a_1 + 2a_2 z + \ldots + n a_n z^{n-1}.$$

If f is differentiable at z_0, then f is continuous at z_0.

Now let f be a complex valued function of a complex variable, so if $z = x + iy$ then $f(z)$ can be written $f(z) = u(x,y) + iv(x,y)$. If the derivative $f'(z)$ of the function exists at a point z_0, then the first partial derivatives of u, v, u_x, u_y, v_x and v_y exists at $z_0 = x_0 + iy_0$ and satisfy the Cauchy Riemann conditions:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and}$$

$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. Conversely if u, v, u_x, u_y, v_x, v_y all exist in a neighborhood of z_0, are continuous at z_0, and satisfy the Cauchy Riemann conditions at z_0, then $f'(z_0)$ exists. In conclusion, if f is analytic at z_0 (or on the open set A), then u_x, u_y, v_x, v_y exist and satisfy the Cauchy Riemann conditions in some neighborhood at z_0 (or on A); conversely if u, v, u_x, u_y, v_x, v_y are continuous and satisfy

the Cauchy Riemann Conditions in some neighborhood of z_0 (or on A), then f is analytic at z_0 (or on A).

Example

Assume $f(z) = u(x,y)$ is a real valued for all z, hence $v(x,y) = 0$. If f is analytic in the domain D, then f is constant on D. To see this, u and v satisfy the Cauchy Riemann conditions on D. Thus $u_x = v_y$ and $u_y = -v_x$. Since $v(x,y) = 0$, then $v_x = v_y = 0$ and so $u_x = u_y = 0$ for all (x,y) in D. This implies that u is constant on D. Hence, there is a complex number c such that $f(z) = u(x,y) = c$ for all z in D.

Sequences and series of complex numbers are very important in analytic function theory. A sequence of complex numbers $\{z_n\}$ is said to converge to z if and only if for each $\varepsilon > 0$, there exists $k \in N$ such that $n > k$ implies $|z_n - z| < \varepsilon$. This is written $\lim_{n\to\infty} z_n = z$. If $z_n = x_n + iy_n$ and $z = x + iy$, then $\{z_n\}$ converges to z if and only if the sequences of real numbers $\{x_n\}$ and $\{y_n\}$converge to x and y, respectively. A sequence $\{z_n\}$is said to be a Cauchy sequence if and only if for each $\varepsilon > 0$ there exists $k \in N$ such that $m \geq n > k$ implies $|z_n - z_m| < \varepsilon$. All convergent sequences are Cauchy sequences and every Cauchy sequence of complex numbers converges to a complex number .

Not every sequence converges, those that do not converge are said to diverge. A special type of divergence is divergence to infinity. A sequence of complex numbers

$\{z_n\}$ is said to diverge to infinity if and only if $\lim_{n\to\infty}|z_n| = +\infty$. This means that for every positive real number r there exists $k \in N$ such that $n > k$ implies $|z_n| > r$.

If all the terms of a sequence $\{z_n\}$ in **C** are added , the result is a series , written $\sum_{n=1}^{\infty} z_n$. This series is said to converge to z if and only if $\lim_{k\to\infty}\sum_{n=0}^{k} z_n = z$. Thus, the sequence of finite sums $S_k = \sum_{n=1}^{k} z_n = z_1 + z_2 + \ldots + z_k$, which is called the k[th] partial sum of the series, converges to z, $\lim_{k\to\infty} S_k = z$. If the series does not converge then it is said to diverge.

Example

The Geometric Series: If z is a complex number, then the series $\sum_{n=0}^{\infty} z^n = z^0 + z^1 + z^2 + \ldots = 1 + z + z^2 + \ldots$ is called a geometric series. This converges to $\frac{1}{1-z}$ if $|z| < 1$ and it diverges if $|z| \geq 1$. Also the series $\sum_{n=1}^{\infty}|z^n| = \sum_{n=1}^{\infty}|z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$.

Next, consider a sequence of complex valued functions $f_1(z), f_2(z), f_3(z), \ldots$ all defined on the domain D. If $D_1 \subset D$ and for each $z \in D_1$, $\lim_{n\to\infty} f_n(z) = f(z)$, then f is a function defined on D_1 and the sequence $\{f_n\}$ is said to converge pointwise to f on D_1.

The series of functions $\sum_{n=1}^{\infty} f_n$ is said to converge pointwise to the function g on D_1 if and only if $\sum_{n=1}^{\infty} f_n(z)$ converges to g(z) for $z \in D_1$.

If $\{a_n\}$ is a sequence of complex numbers and z_0 is a complex number, then $\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$ is called a power series centered at z_0 or just a power series. If $D = \left\{ z \;\middle|\; \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ converges} \right\}$, then the power series defines a function with domain D, namely $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ for $z \in D$.

Notes

1) $z_0 \in D$ since if $z = z_0$, then $(z-z_0)^n = 0^n = 0$ for $n \geq 1$. Thus the series

$$f(z_0) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 \text{ converges.}$$

2) Let $z_0 = 0$ and $a_n = 1$, then the series $\sum_{n=0}^{\infty} z^n$ is a geometric series which converges to $\frac{1}{1-z}$ for $|z| < 1$. Thus $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for z in $D = N_1(0)$

.

3) If $f_n(z) = \sum_{k=0}^{n} a_k (z-z_0)^k$, then

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n = f(z).$$

Theorem 2.4

Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be power series. Then either

(1) the power series converges for every z in **C** or

(2) there exists a real number $R \geq 0$ such that if $|z - z_0| < R$, the series converges and if $|z - z_0| > R$, the series diverges. When $R = 0$, the series diverges for all $z \neq z_0$.

The number R in (2) is called the radius of convergence of the power series. In (1) the radius of convergence is ∞. The series is said to have a positive radius of convergence when $R > 0$, which includes $R = \infty$.

Theorem 2.5

If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)$ has a radius of convergence $R > 0$, then f is continuous on $N_R(z_0)$ and differentiable on $N_R(z_0)$. Moreover, $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ where this power series also has radius of convergence R.

Corollary 2.1

If $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ has radius of convergence $R > 0$, then f is infinitely differentiable in $N_R(z_0)$. Thus $f', f'', f^{(3)}, \ldots, f^{(n)}, \ldots$ all exist in $N_R(z_0)$.

Theorem 2.6

Let $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ be a power series with radius of convergence R > 0.

Then $a_n=\dfrac{f^{(n)}(z_0)}{n!}$ n = 1,2,3,....

Theorem 2.7

Let f be analytic on $N_r(z_0)$ where r > 0. Then f is infinitely differentiable on $N_r(z_0)$ and $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)$ for all z $\in N_r(z_0)$, where $a_n=\dfrac{f^{(n)}(z_0)}{n!}$. The radius for this power series is R ≥ r. This is called the power series representation of f on $N_r(z_0)$.

To summarize these results; If f is analytic at z_0, then $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ where the radius of convergence is r > 0. Thus f is infinitely differentiable on $N_r(z_0)$ and $a_n=\dfrac{f^{(n)}(z_0)}{n!}$. Conversely, if $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ has positive radius of convergence, then f is analytic at z_0.

CHAPTER 3

PROPERITIES OF ANALYTIC FUNCTIONS

The Jordan Curve Theorem 3.1

Let $\gamma:[a,b]\to C$ be a simple closed curve in D, then γ separates **C** into two disjoint domains, one is bounded, denoted by Int(γ) = interior of γ, and the other is unbounded, denoted by Ext(γ) = exterior of γ. A domain D in **C** is simply connected if and only if for every simple closed curve $\gamma:[a,b]\to D$ in D, Int(γ) ⊂ D. This condition says that if every point on the curve is in D, then so is every point inside the curve. Basically, a simple connected domain contains no holes.

Cauchy-Goursat Theorem 3.2

Let D be a simply connected domain and f: D → C be analytic on D. Then for any closed rectifiable curve γ:[a,b] → D in D, $\int_{\gamma} fdz = 0$.

The orientation of a simple closed curve γ describes the direction of the point γ(t) as t moves from a to b. For example the curve $\gamma(t)=e^{it}$, $0\le t\le 2\pi$, describes the unit circle where the direction is counterclockwise. Counterclockwise is defined to be a positive orientation while clockwise means a negative orientation. So this curve is

positively oriented or γ has positive orientation. The curve $\gamma(t)=e^{-it}, 0 \le t \le 2\pi$, defines a circle on which the point $\gamma(t)$ moves in a clockwise direction, so this is a negatively oriented curve.

In general, for complicated curves, clockwise and counterclockwise may be ambiguous. As $\gamma(t)$ moves around the simple closed curve in the direction of increasing values of t, if the interior lies to the left of the point, the curve is positively oriented, otherwise it is negatively oriented.

Cauchy's Integral Formulas 3.3

Let γ be a simple closed positively oriented rectifiable curve and $z_0 \in Int(\gamma)$. If f is analytic on $Tr(\gamma) \cup Int(\gamma)$, then $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz$. Moreover, for each n = 1,2,3,… $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-z_0)^{n+1}}dz$ where $f^{(n)}(z_0)$ the n^{th} derivative of f at z_0.

Proposition 3.1

If f is analytic on $N_r(z_0)$ such that $f(z) = 0$ for $|z-z_0| < r_1$ where $0 < r_1 < r$, then $f(z) = 0$ for $z \in N_r(z_0)$.

Proof

Let $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \ldots$ be the power series representation of f on $N_r(z_0)$, where $a_n = \frac{f^{(n)}(z_0)}{n!}$. Since f is identically equal to 0 on $N_{r_1}(z_0)$, then so are all the derivatives of f. In particular, $a_n = \frac{f^{(n)}(z_0)}{n!} = 0$

for n = 0, 1, 2, …. Thus for z in $N_r(z_0)$, $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \ldots = 0$.

Q.E.D.

Proposition 3.2

Let f be analytic on the domain D. If there exists a neighborhood $N_0 = N_r(z_0)$ contained in D such that $f(z) = 0$ for $z \in N_0$, then $f(z) = 0$ for all $z \in D$.

Proof

Fix $\hat{z} \in D$ and let $\gamma: [a,b] \to D$ be a curve in D from $z_0 = \gamma(a)$ to $\hat{z} = \gamma(b)$. Using the fact that D is an open set and $Tr(\gamma)$ is a compact set in D, a finite sequence of points $z_0, z_1, z_2, \ldots, z_k = \hat{z}$ and neighborhoods $N_0, N_1, N_2, \ldots, N_k$ can be constructed, where N_i is a neighborhood of z_i such that $N_i \subset D$ and $z_i \in N_{i-1} \cap N_i$.

Since f is analytic on D, f is analytic on each N_i. Moreover, $f \equiv 0$ on N_0. But $z_1 \in N_0 \cap N_1$ and so there is a small neighborhood $N_\delta(z_1)$ of z_1 contained in both N_0 and N_1.

Since $f(z) = 0$ on N_0 then this is true on the subset $N_\delta(z_1)$. Thus $f(z) = 0$ for $|z - z_0| < \delta$ and so $f(z) = 0$ for $z \in N_1$ by the previous proposition. Similar arguments shows $f \equiv 0$ on N_2, N_3, and so forth.

In fact this argument can be extended by induction to show $f(z) = 0$ on N_k. Since N_k is a neighborhood of $z_k = \hat{z}$, $f(\hat{z}) = 0$. Consequently, $f(\hat{z}) = 0$ for any $\hat{z} \in D$.

Q.E.D.

Corollary 3.1

Let f be analytic on the domain D. If there exists a neighborhood $N_r(z_0)$ in D such that f is constant on this neighborhood, then f is constant on D.

Proof

Let $f(z) = M$ for every $z \in N_r(z_0)$. Define $g(z) = f(z) - M$. Then g is also analytic on D and $g(z) = 0$ for $z \in N_r(z_0)$. An application of the previous result to g shows that $g(z) = 0$ for every $z \in D$. Thus $f(z) = M$ for every $z \in D$.

Q.E.D.

Corollary 3.2

Let f and g be analytic on D. If there exists a neighborhood $N_0 = N_r(z_0)$ contained in D such that $f(z) = g(z)$ for $z \in N_0$, then $f(z) = g(z)$ for all $z \in D$.

Proof

Let $h(z) = f(z) - g(z)$. Then h is analytic on D such that $h(z) = 0$ on N_0. Thus, $h(z) = 0$ for $z \in D$ by Proposition 3.2. Consequently, $f(z) = g(z)$ on D.

Q.E.D.

Lemma 3.1

If $f = u + iv$ and $\overline{f} = u - iv$ are both analytic on a domain D, then f must be constant on D.

Proof

Assume f and $\overline{f}$ are both analytic, and apply the Cauchy Riemann Conditions to f, $u_x = v_y$ and $u_y = -v_x$; and to $\overline{f}$, $u_x = -v_y$ and $u_y = -(-v_x) = v_x$. Hence $u_x = v_y = -u_x$ which implies $u_x = 0$ on D. Similarly, $u_y = -v_x = -u_y$ implies, $u_y = 0$ on D. Since $u_x = u_y = 0$, then u is constant on D. Additionally, $v_x = u_y = 0$ and $v_y = u_x = 0$, and so v is also constant on D. Hence, $f = u + iv$ must be constant on D.

Q.E.D.

Lemma 3.2

If f is analytic on a domain D such that $|f(z)|$ is constant on D, then $f(z)$ is constant on D.

Proof

If $|f(z)| = 0$ for all $z \in D$, then $f(z) = 0$ for $z \in D$ and the lemma is proved. Now assume there exists $M \neq 0$ such that $|f(z)| = M$ for all $z \in D$.

Thus, $|f(z)|^2 = f(z).\overline{f(z)} = M^2$ and so $\overline{f(z)} = \dfrac{M^2}{f(z)}$. Since $|f(z)| = M \neq 0$, then $f(z) \neq 0$ for all $z \in D$ and so $\overline{f(z)} = \dfrac{M^2}{f(z)}$ is analytic on D. Since f and $\overline{f}$ are both analytic on D, then f must be a constant by the previous lemma.

Q.E.D.

3.4 Theorem (Cauchy's Inequality)

Let f be analytic inside and on the circle $C_r(z_0)$ and $|f(z)| \le M$ for all $z \in C_r(z_0)$,

then $\left|f^{(n)}(z_0)\right| \le \dfrac{n!M}{r^n}$ for n = 0, 1, 2, 3, ….

Proof

Let $\gamma_r = z_0 + re^{it}$, $0 \le t \le 2\pi$ be the simple closed positively oriented curve whose trace is $C_r(z_0)$. Then according to Cauchy's Integral Formula,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z\ )}{(z-z_0)^{n+1}}dz\,.$$

But for $z \in Tr(\gamma_r)$, $|f(z_0)| \le M$ and $|z - z_0| = r$, thus $\left|\dfrac{f(z_0)}{(z-z_0)^{n+1}}\right| \le \dfrac{M}{r^{n+1}}$. Accordingly,

$$\left|f^{n}(z_0)\right| = \left|\frac{n!}{2\pi i}\right|\left|\int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n=1}}dz\right| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}}L(\gamma_r)$$

$$= \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r = \frac{n!M}{r^n}\,.$$

Q.E.D.

The next theorem, Liouville's Theorem, states that if f is analytic on **C** and is also a bounded function on **C**, then f must be constant on **C**. Notice that this is not true for real valued functions; for instance, $f(x) = \sin(x)$, $x \in$ **R**, is differentiable on **R** and bounded on **R.** To prove this theorem the following lemma is necessary:

Lemma 3.3

If f is defined on a domain D such that $f'(z) = 0$ for all z ∈ D, then f is constant on D.

Proof

Let z_0, z_1 ∈ D, and let γ be any rectifiable curve from z_0 to z_1, which exists because D is path connected . By the Fundamental Theorem of Calculus,

$f(z_1) - f(z_0) = \int_\gamma f'(z)dz = 0$, because $f'(z) = 0$ for all z ∈ Tr(γ) ⊂ D. Thus for all z_1 ∈ D, $f(z_1) = f(z_0)$, which proves that f is constant on D.

Q.E.D.

Liouville's Theorem 3.5

Let f be analytic on **C** such that f is bounded on **C**. Then f is a constant function.

Proof

By the previous lemma it is enough to show that $f'(z_0) = 0$ for all z_0 ∈ C. Since f is bounded , there exists M > 0 such that $|f(z)| \le M$ for all z. Fix z_0 ∈ **C** and let r > 0 be any positive number, then f is analytic on $C_r(z_0) \cup Int(C_r(z_0))$.

For any ε > 0 , let r = $\frac{2M}{\varepsilon}$. According to Cauchy's Inequality

$|f'(z_0)| \le \frac{M}{r} = \frac{M}{\frac{2M}{\varepsilon}} = \frac{M\varepsilon}{2M} = \frac{\varepsilon}{2} < \varepsilon$. Consequently, $|f'(z_0)| < \varepsilon$ for any ε > 0, which proves $f'(z_0) = 0$.

Q.E.D.

Proof

Let z_0 , $z_1 \in$ D, and let γ be any rectifiable curve from z_0 to z_1, which exists because D is path connected . By the Fundamental Theorem of Calculus,

$$f(z_1) - f(z_0) = \int_\gamma f'(z)dz = 0 \text{ , because } f'(z) = 0 \text{ for all } z \in \mathrm{Tr}(\gamma) \subset D.$$

Thus for all $z_1 \in$ D, $f(z_1) = f(z_0)$, which proves that f is constant on D.

Q.E.D.

Liouville's Theorem 3.5

Let f be analytic on **C** such that f is bounded on **C**. Then f is a constant function.

Proof

By the previous lemma it is enough to show that $f'(z_0) = 0$ for all $z_0 \in$ C. Since f is bounded, there exists M > 0 such that $|f(z)| \le M$ for all z. Fix $z_0 \in$ **C** and let r > 0 be any positive number, then f is analytic on $C_r(z_0) \cup Int(C_r(z_0))$.

For any $\varepsilon > 0$, let $r = \dfrac{2M}{\varepsilon}$. According to Cauchy's Inequality

$|f'(z_0)| \le \dfrac{M}{r} = \dfrac{M}{\frac{2M}{\varepsilon}} = \dfrac{M\varepsilon}{2M} = \dfrac{\varepsilon}{2} < \varepsilon$. Consequently, $|f'(z_0)| < \varepsilon$ for any $\varepsilon > 0$, which proves $f'(z_0) = 0$.

Q.E.D.

CHAPTER 4

THE THREE IMPORTANT THEOREMS

Maximum Modulus Principle 4.1

Let f be analytic at z_0 such that f is not constant on any neighborhood of z_0. Then for $\varepsilon > 0$ there exists $z \in N_\varepsilon(z_0)$ such that $|f(z)| > |f(z_0)|$. Hence every neighborhood of z_0 contains a point z such that the modulus of f at z is greater than the modulus of f at z_0. Thus, z_0 cannot be a local maximum point of $|f(z)|$. Note that this is not true for real valued functions, for example: If $f(x) = \cos(x)$, then for $x \in (-\pi,\pi) = N_\pi(0)$, $|\cos(x)| \leq 1 = \cos(0)$.

Proof

Let ε be an arbitrary positive number. Since f is analytic at z_0, there exists $\delta > 0$ such that f is analytic on $N_\delta(z_0)$. If r_1 = smaller of $\delta/2$ and $\varepsilon/2$, then f is analytic on $B_{r_1}(z_0) \subset N_\delta(z_0)$, f is not constant on $N_{r_1}(z_0) \subset B_{r_1}(z_0)$, and $B_{r_1}(z_0) \subset N_\varepsilon(z_0)$. If it is shown that there exists $z \in N_{r_1}(z_0) \subset N_\varepsilon(z_0)$ such that $|f(z)| > |f(z_0)|$, the theorem will

be proved. Assume no such z exists, thus $|f(z)| \le |f(z_0)|$ for all $z \in N_{r_1}(z_0)$. Now $|f(z)|$ is not constant on $N_r(z_0)$ because $f(z)$ is not constant on $N_{r_1}(z_0)$ (by Lemma 3.2).

Since $|f(z)| \le |f(z_0)|$, there exists $z_1 \in N_{r_1}(z_0)$ such that $|f(z_1)| < |f(z_0)|$. If $r = |z_r - z_0| < r_1$, then $z_1 \in C_r(z_0) \subset N_{r_1}(z_0)$. Let $\gamma_r(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$, hence γ_r is the simple closed positively oriented curve with trace $C_r(z_0)$. Since $z_1 \in C_r(z_0) = \text{Tr}(\gamma_r)$, then $\gamma_r(t_1) = z_1$ for some $t_1 \in [0,2\pi]$. If $\gamma(0) = \gamma(2\pi) = z_1$, a second point z_2 on the circle close to z_1 can be chosen $|f(z_2)| < |f(z_0)|$. Hence, it can be assumed that $t_1 \in (0,2\pi)$.

But $|(f(\gamma_r(t_1))| = |f(z_1)| < |f(z_0)|$ where $f \circ \gamma_r$ is continuous on $[0,2\pi]$; consequently, there exists $b < |f(z_0)|$ and $t_2 < t_3$ in $[0,2\pi]$ such that $t_1 \in [t_2,t_3]$ and $|f(\gamma(t))| \le b$ for $t \in [t_2,t_3]$.

Let $\gamma_2 = \gamma_r / [t_2,t_3]$ and since $t_2 < t_3$, then $\text{Tr}(\gamma_2)$ is an arc of positive length contained on the circle $C_r(z_0)$. Thus $L(\gamma_2) > 0$.

Now if $\gamma_1 = \gamma_r / [0,t_2]$ and $\gamma_3 = \gamma_r / [t_3,2\pi]$, then $\gamma_r = \gamma_1 \# \gamma_2 \# \gamma_3$, and so

$$\int_{\gamma} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \text{ and } L(\gamma) = L(\gamma_1) + L(\gamma_2) + L(\gamma_3).$$

For i = 1 or 3, $z \in \text{Tr}(\gamma_i) \subset \text{Tr}(\gamma_r)$ implies $|f(z)| \le |f(z_0)|$ and $|z - z_0| = r$, thus

$$\left|\frac{f(z)}{z - z_0}\right| \le \left|\frac{f(z_0)}{r}\right|.$$

Consequently, $\left|\frac{1}{2\pi i}\int_{\gamma_i}\frac{f(z)}{z-z_0}dz\right| = \frac{1}{2\pi}\left|\int_{\gamma_i}\frac{f(z)}{z-z_0}dz\right| \le \frac{1}{2\pi}\frac{|f(z_0)|}{r}L(\gamma_i) = \frac{|f(z_0)|}{2\pi r}L(\gamma_i)$

for γ_1 and γ_3.

For $z \in Tr(\gamma_2)$, $|f(z)| \le b < |f(z_0)|$ and so $\left|\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(z)}{z-z_0}dz\right| \le \frac{b}{2\pi r}L(\gamma_2) <$

$\frac{|f(z_0)|}{2\pi r}L(\gamma_2)$, because $L(\gamma_2) > 0$ and $b < |f(z_0)|$. By Cauchy's Integral Formula

$$|f(z_0)| = \left|\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(z)}{z-z_0}dz\right| = \left|\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(z)}{z-z_0}dz + \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(z)}{z-z_0}dz + \frac{1}{2\pi i}\int_{\gamma_3}\frac{f(z)}{z-z_0}dz\right| \le$$

$$\left|\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(z)}{z-z_0}dz\right| + \left|\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(z)}{z-z_0}dz\right| + \left|\frac{1}{2\pi i}\int_{\gamma_3}\frac{f(z)}{z-z_0}dz\right| \le \frac{|f(z_0)|}{2\pi r}L(\gamma_1) + \frac{b}{2\pi r}L(\gamma_2) + \frac{|f(z_0)|}{2\pi r}L(\gamma_3)$$

$$< \frac{|f(z_0)|}{2\pi r}L(\gamma_1) + \frac{|f(z_0)|}{2\pi r}L(\gamma_2) + \frac{|f(z_0)|}{2\pi r}L(\gamma_3)$$

$= \frac{|f(z_0)|}{2\pi r}(L(\gamma) + L(\gamma_2) + L(\gamma_3)) = \frac{f(z_0)}{2\pi r}L(\gamma_r) = \frac{|f(z_0)|}{2\pi r}2\pi r = |f(z_0)|$. Hence, the

assumption that $|f(z)| \le |f(z_0)|$ on $N_r(z_0)$ implies the contradiction $|f(z_0)| < |f(z_0)|$.

Thus $|f(z)| > |f(z_0)|$ for some $z \in N_{r_1}(z_0) \subset N_\varepsilon(z_0)$.

Q.E.D.

Corollary 4.1

If f is analytic on a domain D containing $B_r(z_0)$ such that $|f(z)| \leq M$ on $C_r(z_0)$ for some constant M, then $|f(z)| \leq M$ on $B_r(z_0)$. Moreover, either f is constant on $B_r(z_0)$ or $|f(z)| < M$ on $N_r(z_0)$.

Proof

If f is constant on $B_r(z_0)$, then $|f(z)|$ is constant on $B_r(z_0)$and $|f(z)| \leq M$ on $C_r(z_0) \subset B_r(z_0)$. Hence there exists a constant L such that $|f(z)| = L \leq M$ for every $z \in B_r(z_0)$. Next assume f is not constant on $B_r(z_0)$. Thus according to Corollary 3.1, f is not constant on any neighborhood of z_0. Since $B_r(z_0)$ is a compact set, then continuity of f on $B_r(z_0)$ implies there exists $z_1 \in B_r(z_0)$ such that $|f(z)| \leq |f(z_1)|$ for all $z \in B_r(z_0)$. If $z_1 \in N_r(z_0)$, then there exists $\varepsilon > 0$ such that $N_\varepsilon(z_1) \subset N_r(z_0) \subset B_r(z_0)$.

According to the Maximum Modulus Principle, there exists z_2 in $N_\varepsilon(z_1)$ such that $|f(z_2)| > |f(z_1)|$, which contradicts $|f(z)| \leq |f(z_1)|$ for every z in $B_r(z_0)$. Thus, $z_1 \notin N_r(z_0)$ and so $z_1 \in C_r(z_0)$. Therefore $|f(z)| \leq |f(z_1)| \leq M$ for $z \in B_r(z_0)$. Moreover, for any $z \in N_r(z_0)$, there exists z_2 in $N_r(z_0)$ such that $|f(z_2)| > |f(z)|$ by the Maximum Modulus Principle, as above. But $z_2 \in B_r(z_0)$ and so $|f(z)| < |f(z_2)| \leq M$. Hence, $|f(z)| < M$ for any z in $N_r(z_0)$.

Q.E.D.

Theorem (Schwarz's Lemma) 4.2

Let f be analytic on D = $N_1(0)$ such that $f(0) = 0$ and $|f(z)| \le 1$ on D. Then $|f(z)| \le |z|$ for z $\in$ D. Moreover, either

1) $|f(z)| < |z|$ for z $\in$ D and z $\neq$ 0, or

2) there exists a real constant ϕ such that $f(z) = ze^{i\phi}$ for z $\in$ D.

Proof

Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$ be the power series representation of f in D. Since $f(0) = 0$, then $a_0 = 0$ and so $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$.

Define $g(z) = \dfrac{f(z)}{z}$ and $h(z) = a_1 + a_2 z + a_3 z^2 + \ldots$. Note that the series for f, $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots = z(a_1 + a_2 z + a_3 z^2 + \ldots) = zh(z)$, converges for $|z| < 1$ and, thus so does the series for h. Consequently, h is analytic on $N_1(0)$ = D and $h(z) = g(z)$ for z $\neq$ 0. Consider any positive R < 1, $C_R = \{z \| z| = R\}$ and for z $\in$ C_R,

$|h(z)| = |g(z)| = \left|\dfrac{f(z)}{z}\right| = \dfrac{|f(z)|}{|z|} \le \dfrac{1}{|z|} = \dfrac{1}{R}$. Consequently, h is analytic on $B_R(0) \subset$ D such that $|h(z)| \le \dfrac{1}{R}$ on C_R. According to Corollary 4.1 following the Maximum Modulus Principle, $|h(z)| \le \dfrac{1}{R}$ on $B_R(0)$.

If $z_0 \neq 0$ is a point in $N_1(0)$ and R is chosen such that $|z_0| \le R < 1$, then $z_0 \in B_R(0)$ and so $|h(z_0)| \le \dfrac{1}{R}$. But $h(z_0) = g(z_0) = \dfrac{f(z_0)}{z_0}$ and so $\left|\dfrac{f(z_0)}{z_0}\right| \le \dfrac{1}{R}$.

Consequently, $\left|f(z_0)\right| \le \frac{|z_0|}{R}$.

Taking the limit as R approaches 1, $\left|f(z_0)\right| \le \lim_{R\to 1} \frac{|z_0|}{R} = |z_0|$. Therefore, since $f(0) = 0$, $\left|f(z_0)\right| \le |z_0|$ for all z_0 in $N_1(0)$.

To prove that either,

1) $|f(z)| < |z|$ for all $z \neq 0$ in $N_1(0)$,

or

2) there exists a real constant ϕ such that $f(z) = e^{i\phi}z$

for all $z \in N_1(0)$, assume (1) is false. Hence there exists z^* such that $\left|f(z^*)\right| = |z^*|$ and $0 < |z^*| < 1$. Thus, $\frac{f(z^*)}{z^*} \in C_1$ (C_1 is the unit circle). Since $C_1 = \{e^{it} | t \in R\}$, then there is a real number ϕ such that $\frac{f(z^*)}{z^*} = e^{i\phi}$. Let g and h be as in the first part of this proof. Then $h(z) = \frac{f(z)}{z}$ for $0 < |z| < 1$, and h is analytic on $N_1(0)$. Thus $|h(z^*)| = 1$, and $h(z^*) = e^{i\phi}$. By the first part of this proof, $|h(z)| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} \le 1$ for all $z \neq 0$ in $N_1(0)$. Hence,

$|h(0)| = \lim_{z\to 0} |h(z)| \le 1.$

Fix $z_0 \in N_1(0)$ and $z_0 \neq 0$, so $0 < |z_0| < 1$. Let r be a number such that $|z_0| \leq r < 1$, and $|z^*| < r < 1$. Then h is analytic on $B_r(0)$, where $z_0 \in B_r(0)$ and $z^* \in N_r(0)$. Now $|h(z)| \leq 1 = |h(z^*)|$ for all $z \in B_r(0) \subset N_1(0)$ and since $z^* \in N_r(0)$, then h must be constant on $B_r(0)$ by the Corollary following the Maximum Modulus Principle. Consequently, $h(z_0) = h(z^*) = e^{i\phi}$. This argument is valid for every $z_0 \in N_1(0)$, so for every $z \in N_1(0)$, $z \neq 0$, $\dfrac{f(z)}{z} = h(z) = e^{i\phi}$. Thus $f(z) = ze^{i\phi}$ for $z \neq 0$ in $N_1(0)$. Since $f(z) = 0$,then $f(z) = ze^{i\phi}$ for every $z \in N_1(0)$.

Q.E.D.

The following result is needed in the proofs on both The Open – Mapping Theorem, and the Inverse Function Theorem.

Theorem 4.3

Suppose f is analytic on $N_r(z_0)$ such that $f'(z_0) \neq 0$ and let $w_0 = f(z_0)$. Then there exists $r_1, r_2 > 0$ with $r_1 < r$ and a function g: $N_{r_2}(w_0) \to B_{r_1}(z_0) \subset N_r(z_0)$ such that

(1) f is one-to-one on $B_{r_1}(z_0)$,

(2) $N_{r_2}(w_0) \subset f(B_{r_1}(z_0)) \subset f(N_r(z_0))$,

(3) $f(g(w)) = w$ for all $w \in N_{r_2}(w_0)$,

(4) g is the inverse of f on $N_{r_2}(w_0)$,

(5) $f'(z_0) \neq 0$ for $z \in B_{r_1}(z_0)$,

(6) g is analytic on $N_{r_2}(w_0)$ such that $g'(w) = \dfrac{1}{f'(g(w))}$ for $w \in N_{r_2}(w_0)$.

Proof

Since f is analytic on $N_r(z_0)$, f' is continuous and $f'(z_0) \neq 0$. Hence, for $\varepsilon = \frac{1}{2}|f'(z_0)| > 0$, there exists $0 < \delta \leq r$ such that $|z - z_0| < \delta$ implies

$$|f'(z) - f'(z_0)| < \epsilon = \frac{1}{2}|f'(z_0)| \quad \ldots(1)^*$$

If $0 < r_1 < \delta$, then $(1)^*$ holds for $z \in B_{r_1}(z_0)$ and f is analytic on $B_{r_1}(z_0) \subset N_r(z_0)$.

Define $R(z) = f(z) - f(z_0) - (z - z_0)f'(z_0)$. ….$(2)^*$, R is defined and continuous on $B_{r_1}(z_0)$. If $z_1, z_2 \in B_{r_1}(z_0)$,

$$\begin{aligned} R(z_1) - R(z_2) &= f(z_1) - f(z_0) - (z_1 - z_0)f'(z_0) - [f(z_2) - f(z_0) - (z_2 - z_0)f'(z_0)] \\ &= f(z_1) - f(z_2) - z_1 f'(z_0) + z_2 f'(z_0) \\ &= f(z_1) - f(z_2) - (z_1 - z_2)f'(z_0). \end{aligned}$$

Consider γ the straight line segment from z_2 to z_1 which is contained in $B_{r_1}(z_0)$.

Then by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_\gamma (f'(z) - f'(z_0))dz = (f(z) - f'(z_0)z)\Big|_{z_2}^{z_1} &= f(z_1) - f'(z_0)z_1 - f(z_2) + f'(z_0)z_2 \\ &= f(z_1) - f(z_2) - f'(z_0)(z_1 - z_2) \\ &= R(z_1) - R(z_2). \end{aligned}$$

Therefore $R(z_1) - R(z_2) = \int_\gamma (f'(z) - f'(z_0))dz$. Consequently,

$$|R(z_1) - R(z_2)| = \left|\int_\gamma (f'(z) - f'(z_0)dz\right| \leq \frac{1}{2}|f'(z_0)|L(\gamma) = \frac{1}{2}|f'(z_0)||z_1 - z_2| \ (1)^*.$$

Therefore if $z_1, z_2 \in B_{r_1}(z_0)$, then $|R(z_1) - R(z_2)| \le \left|\frac{1}{2}\right| |f'(z_0)| |z_1 - z_2|$. (3)*.

Note that $R(z_0) = f(z_0) - f(z_0) - (z_0 - z_0) f'(z_0) = 0$. Now let

$$a = \frac{1}{f'(z_0)} \quad \text{and} \quad r_2 = \frac{1}{2}|f'(z_0)| r_1 > 0.$$

A sequence of functions $g_1, g_2, g_3, \ldots$ will be defined inductively, such that:

(1) g_n is continuous on $N_{r_2}(w_0)$, and for $w \in N_{r_2}(w_0)$,

(2) $|g_n(w) - g_{n-1}(w)| < 2^{-n} r_1$ if $n > 1$,

(3) $|g_n(w) - z_0| < (1 - 2^{-n}) r_1$,

(4) $g_{n+1}(w) = z_0 + a(w - w_0) - aR(g_n(w))$.

Define $g_1(w) = z_0 + a(w - w_0)$ for $w \in N_{r_2}(w_0)$. Clearly g_1 satisfies (1).

If $w \in N_{r_2}(w_0)$, $|g_1(w) - z_0| = |a(w - w_0)| = |a||w - w_0| < |a| r_2 = \frac{1}{|f'(z_0)|}\frac{1}{2}|f'(z_0)| r_1$

$= \frac{1}{2} r_1 = (1 - 2^{-1}) r_1$. Thus, g_1 satisfies (3).

Since $|g_1(w) - z_0| < \frac{1}{2} r_1 < r_1$, then $g_1(w) \in N_{r_1}(z_0)$ and so $R(g_1(w))$ is defined and continuous in w. In other words, g_1 is continuous on $N_{r_2}(w_0)$ and R is continuous on $g_1(N_{r_2}(w_0)) \subset N_{r_1}(z_0)$, hence, the composition $R(g_1(w))$ is continuous on $N_{r_2}(w_0)$.

Now define $g_2(w) = z_0 + a(w - w_0) - aR(g_1(w))$ for $w \in N_{r_2}(w_0)$. By the previous comment g_2 is continuous on $N_{r_2}(w_0)$. Moreover,

$$|g_2(w) - g_1(w)| = |z_0 + a(w - w_0) - aR(g_1(w)) - (z_0 + a(w - w_0))|$$

$$= |aR(g_1(w))|.$$

Since $R(z_0) = 0$, then by(3^*) $|g_2(w) - g_1(w)| = |aR(g_1(w))| = |aR(g_1(w)) - aR(z_0)|$

$\leq |a|\frac{1}{2}|f'(z_0)||g_1(w) - z_0| = \frac{1}{2}|g_1(w) - z_0| < \frac{1}{2}(1-2^{-1})r_1 = \frac{1}{2}\left(\frac{1}{2}\right)r_1 = \frac{1}{4}r_1$. The first inequality follows from 3^* and the second one from (3).

Thus, $|g_2(w) - g_1(w)| < 2^{-2}r_1$ and so g_1 satisfies (2).

$|g_2(w) - z_0| \leq |g_2(w) - g_1(w)| + |g_1(w) - z_0| < 2^{-2}r_1 + (1-2^{-1})r_1 =$

$\left(\frac{1}{4} + \frac{1}{2}\right)r_1 = \left(1 - \frac{1}{4}\right)r_1 = (1 - 2^{-2})r_1$. Hence, $|g_2(w) - z_0| < (1 - 2^{-2})r_1$ and so g_2 also satisfies (3). Furthermore, g_2 satisfies (4) by definition.

Inductively assume $g_1, g_2, \ldots, g_k$ have been defined so that (1) to (4) are true for $n \leq k$. Define $g_{k+1}(w) = z_0 + a(w - w_0) - aR(g_k(w))$. First $g_k(w)$ is defined and continuous on $N_{r_2}(w_0)$ because $g_k(N_{r_2}(w_0)) \subset N_{r_1}(z_0)$ by (3); in the other words,

$|g_k(w) - z_0| < (1 - 2^{-k})r_1 < r_1$.

$$|g_{k+1}(w) - g_k(w)| = |z_0 + a(w - w_0) - aR(g_k(w)) - [z_0 + a(w - w_0) - aR(g_{k-1}(w))]|$$

$$= |aR(g_{k-1}(w)) - aR(g_k(w))|$$

$$= |a||R(g_{k-1}(w)) - R(g_k(w))|$$

$$\leq |a|\frac{1}{2}|f'(z_0)||g_{k-1}(w) - g_k(w)| \quad \text{(by } 3^*\text{)}$$

$$= \frac{1}{2}\left|g_{k-1}(w) - g_k(w)\right|$$

$$< \frac{1}{2}\left(2^{-k}\right)r_1 = 2^{-(k+1)}r_1$$ by (2) for n = k. This proves (2) for n = k+1.

Finally, $\left|g_{k+1}(w) - z_0\right| \le \left|g_{k+1}(w) - g_k(w)\right| + \left|g_k(w) - z_0\right| < 2^{-(k+1)}r_1 + (1-2^{-k})r_1$

$= \left(1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}\right)r_1 = \left(1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}}\right)r_1 = \left(1 - \frac{1}{2^{k+1}}\right)r_1$. This establishes (3) for n = k+1.

Again, (4) is true by definition. This completes the inductive step.

For any $w \in N_{r_2}(w_0)$ and for n > m ,

$\left|g_n(w) - g_m(w)\right| = \left|g_n(w) - g_{n-1}(w) + g_{n-1}(w) - g_{n-2}(w) + g_{n-2}(w) \ldots + g_{m+1}(w) - g_m(w)\right| \le$

$\left|g_n(w) - g_{n-1}(w)\right| + \left|g_{n-1}(w) - g_{n-2}(w)\right| + \ldots + \left|g_{m+1}(w) - g_m(w)\right| < 2^{-n}r_1 + 2^{-(n-1)}r_1 + \ldots + 2^{-(m+1)}r_1$

$= \sum_{k=m+1}^{n} 2^{-k}$. Since the geometric series $\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} 2^{-k}$ converges, then for ε > 0 there exists N such that n ≥ m ≥ N implies $\sum_{k=m+1}^{n} 2^{-k} < \frac{\varepsilon}{r_1}$. Hence,

$\left|g_n(w) - g_m(w)\right| < \sum_{k=m+1}^{n} 2^{-k} r_1 < \varepsilon$ for every w in $N_{r_2}(w_0)$ and for n > m ≥ N. This proves that $\{g_n(w)\}$ is a Cauchy sequence of complex numbers which, therefore, converges to a complex number denoted by g(w). Consequently, g is a function defined on $N_{r_2}(w_0)$ such that $g(w) = \lim_{n\to\infty} g_n(w)$.

Again for $w \in N_{r_2}(w_0)$, $\left|g_n(w) - z_0\right| < (1-2^{-n})r_1 < r_1$ (n = 1,2,3,…) and since $\lim_{n\to\infty} g_n(w) = g(w)$, $\left|g(w) - z_0\right| \le r_1$. This proves $g(N_{r_2}(w_0)) \subset B_{r_1}(z_0)$.

Claim 1: g is continuous on $N_{r_2}(w_0)$. Fixing w_1 in $N_{r_2}(w_0)$, let $\varepsilon > 0$ be an arbitrary positive number . By the previous argument there exists N such that $n > m \geq N$ implies

$|g_n(w) - g_m(w)| < \frac{\varepsilon}{4}$ for all $w \in N_{r_2}(w_0)$. Since $\lim_{n\to\infty} g_n(w_1) = g(w_1)$, there exists

$m \geq N$ such that $|g_m(w_1) - g(w_1)| < \frac{\varepsilon}{4}$. By continuity of g_m on $N_{r_2}(w_0)$, there exists $\delta > 0$

such that $|w - w_1| < \delta$ implies $|g_m(w) - g_m(w_1)| < \frac{\varepsilon}{4}$ and $w \in N_{r_2}(w_0)$.

For $w \in N_\delta(w_1)$, $\lim_{n\to\infty} g_n(w) = g(w)$ implies there exists $n > m$ such that

$|g_n(w) - g(w)| < \frac{\varepsilon}{4}$. Therefore, $|g(w) - g(w_1)| \leq |g(w) - g_n(w)| + |g_n(w) - g_m(w)| +$

$$|g_m(w) - g_m(w_1)| + |g_m(w_1) - g(w_1)| <$$

$$\frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Consequently, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|w - w_1| < \delta$ implies $|g(w) - g(w_1)| < \varepsilon$. This proves g is continuous at any $w_1 \in N_{r_2}(w_0)$ and so g is continuous on $N_{r_2}(w_0)$.

To see that $f(g(w)) = w$ for $w \in N_{r_2}(w_0)$;

$$g(w) = \lim_{n\to\infty} g_n(w)$$

$$= \lim_{n\to\infty}(z_0 + a(w - w_0) - aR(g_{n-1}(w)))$$

$$= z_0 + a(w - w_0) - a \lim_{n\to\infty} R(g_{n-1}(w))$$

$$= z_0 + a(w - w_0) - aR(\lim_{n\to\infty} g_{n-1}(w)) \quad \text{(because R is continuous on } B_{r_1}(z_0)\text{)}$$

$$= z_0 + a(w - w_0) - aR(g(w)).$$

Recall that $a = \dfrac{1}{f'(z_0)}$ and $f(z_0) = w_0$, hence

$$g(w) = z_0 + a(w - w_0) - aR(g(w))$$

$$= z_0 + a(w - w_0) - a\left[f(g(w)) - f(z_0) - f'(z_0)(g(w) - z_0)\right]$$

$$= z_0 + a(w - w_0) - af(g(w)) + aw_0 + g(w) - z_0$$

$$= aw - af(g(w)) + g(w).$$

Thus $g(w) = aw - af(gw)) + g(w)$, and since $a \neq 0$, $f(g(w)) = w$.

Consequently $g: N_{r_2}(w_0) \to B_{r_1}(z_0)$ is continuous function such that $f(g(w)) = w$ for $w \in N_{r_2}(w_0)$.

Claim 2: f is one-to-one on $B_{r_1}(z_0)$. Let $z_1, z_2 \in B_{r_1}(z_0)$ such that $f(z_1) = f(z_2)$.

$$R(z_1) - R(z_2) = f(z_1) - f(z_0) - (z_1 - z_0)f'(z_0) - \left[f(z_2) - f(z_0) - (z_2 - z_0)f'(z_0)\right]$$

$$= f(z_1) - f(z_2) - z_1 f'(z_0) + z_2 f'(z_0)$$

$$= -(z_1 - z_2)f'(z_0) \quad \text{since } f(z_1) = f(z_2).$$

Therefore,

$$|R(z_1) - R(z_2)| = |-(z_1 - z_2)f'(z_0)|$$

$$= |z_1 - z_2||f'(z_0)|.$$

By (3*) $\left|R(z_1)-R(z_2)\right| \le \frac{1}{2}\left|f'(z_0)\right|\left|z_1-z_2\right|$ and so

$\left|z_1-z_2\right|\left|f'(z_0)\right| \le \frac{1}{2}\left|z_1-z_2\right|\left|f'(z_0)\right|$. But $f'(z_0)\ne 0$, hence, $\left|z_1-z_2\right|=0$. Thus,

$z_1=z_2$ which proves claim 2.

If $w\in N_{r_2}(w_0)$, then g(w) $\in$ $B_{r_1}(z_0)$ such that f(g(w)) = w. This proves $w\in f(B_{r_1}(z_0))$; therefore, $N_{r_2}(w_0)\subset f(B_{r_1}(z_0))$. Furthermore, *f* is one-to-one on $B_{r_1}(z_0)$, hence,

$$f: B_{r_1}(z_0)\to f(B_{r_1}(z_0)) \text{ has an inverse } f^{-1},$$

$$f^{-1}: f(B_{r_1}(z_0))\to B_{r_1}(z_0)\ .$$

If $w\in N_{r_2}(w_0)\subset f(B_{r_1}(z_0))$, then $f(g(w))=w=f(f^{-1}(w))$. But g(*w*) and $f^{-1}(w)$ are in $B_{r_1}(z_0)$ and *f* is one-to-one on this set, hence $g(w)=f^{-1}(w)$ for $w\in N_{r_2}(w_0)$. Thus g = f^{-1} on $N_{r_2}(w_0)$. Consequently, g is the inverse of *f* on the set $g(N_{r_2}(w_0))\subset B_{r_1}(z_0)$.

Let $z\in B_{r_1}(z_0)$ and by (1*) $\left|f'(z)-f'(z_0)\right| < \frac{1}{2}\left|f'(z_0)\right|$. This implies $f'(z)\ne 0$ because $\left|0-f'(z_0)\right| = \left|f'(z_0)\right| < \frac{1}{2}\left|f'(z_0)\right|$ is impossible. Therefore, $f'(z)\ne 0$ on $B_{r_1}(z_0)$.

Finally to prove g is analytic on $N_{r_2}(w_0)$, fix $w_1\in N_{r_2}(w_0)$ and let $z_1=g(w_1)$. If $w\in N_{r_2}(w_0)$ and $z=g(w)$, then $z_1, z\in B_{r_1}(z_0)\subset N_r(z_0)$. Hence, f is differentiable at z_1 and $f'(z_1)\ne 0$. Furthermore, because g is continuous,

$\lim_{w\to w_1} g(w) = g(w_1)$. In other words, $z\to z_1$ as $w\to w_1$. Also note,

$f(z) = f(g(w)) = w$ and similarly $f(z_1) = w_1$.

Consequently, $\lim\limits_{w \to w_1} \dfrac{w - w_1}{g(w) - g(w_1)} = \lim\limits_{z \to z_1} \dfrac{f(z) - f(z_1)}{z - z_1} = f'(z_1) \neq 0.$

Therefore, $\lim\limits_{w \to w_1} \dfrac{g(w) - g(w_1)}{w - w_1} = \lim\limits_{w \to w_1} \left(\dfrac{w - w_1}{g(w) - g(w_1)} \right)^{-1} = \dfrac{1}{f'(z_1)}.$

Consequently, g is differentiable at w_1 such that $g'(w_1) = \dfrac{1}{f'(z_1)} = \dfrac{1}{f'(g(w_1))}$.

This proves that g is differentiable, and thus analytic, on the open set $N_{r_2}(w_0)$ such that

$g'(w) = \dfrac{1}{f'(g(w))}$.

Q.E.D.

Open-Mapping Theorem 4.4

Let f be analytic on the domain D such that $f'(z) \neq 0$ on D. Then for every open set U $\subset$ D, f(U) is an open set. Moreover, f(D) is a domain. Note: Any continuous function which takes open sets onto open sets is called an open mapping.

Proof

Let U be an open set in D and $w_0 \in f(U)$. Then there exists $z_0 \in$ U such that $f(z_0) = w_0$. Moreover, since U is open, there exists r > 0 such that $N_r(z_0) \subset U \subset D$. Hence f is analytic on $N_r(z_0)$ and $f'(z_0) \neq 0$. According to the previous theorem there exists $r_2 > 0$ such that $N_{r_2}(w_0) \subset f(N_r(z_0)) \subset f(U)$. Therefore, if $w_0 \in f$(U), there exists $r_2 > 0$ such that $N_{r_2}(w_0) \subset f(U)$. This proves f(U) is an open set. Since D is open, then f(D) is open too.

For w_1, w_2 in f(D), there exists $z_1, z_2 \in$ D such that $f(z_1) = w_1$ and $f(z_2) = w_2$.

Because D is a path connected, there exists a path $\gamma:[a,b] \to D$ in D such that

$\gamma(a) = z_1$ and $\gamma(b) = z_2$. Thus $f \circ \gamma:[a,b] \to f(D)$ is a path in f(D) such that

$$(f \circ \gamma)(a) = f(\gamma(a)) = f(z_1) = w_1 \text{ and}$$

$$(f \circ \gamma)(b) = f(\gamma(b)) = f(z_2) = w_2 .$$

This proves f(D) is also a path connected. Hence f(D) is an open path connected set; in other words, f(D) is a domain.

Q.E.D.

Inverse Function Theorem 4.5

Let f : D $\to$ C be analytic and one-to-one on the domain D such that $f'(z) \neq 0$ on D. Then f has an analytic inverse $\hat{f}: f(D) \to D$ such that

$$\hat{f}'(w) = \frac{1}{f'(\hat{f}(w))} .$$

Proof

Since f: D $\to$ f(D) is one-to-one and onto then f has a unique inverse $\hat{f}: f(D) \to D$. Moreover, f(D) is also a domain by the Open Mapping Theorem.

Fix $w_0 \in f(D)$, then there exists $z_0 \in D$ such that $f(z_0) = w_0$. Moreover, there exists r > 0 such that $N_r(z_0) \subset D$. Hence f is an analytic on $N_r(z_0)$ such that $f'(z_0) \neq 0$.

By Theorem 4.3 , there exists $r_2 > 0$ and a function $g: N_{r_2}(w_0) \to N_r(z_0)$ such that f(g(w)) = w for w $\in N_{r_2}(w_0)$. Moreover, g is analytic on $N_{r_2}(w_0)$ with

$$g'(w) = \frac{1}{f'(g(w))} \quad \text{for } w \in N_{r_2}(w_0) .$$

But $\hat{f}$ is the inverse of f on D, thus $f(\hat{f}(w)) = w$ for $w \in N_{r_2}(w_0)$.

Consequently, $f(g(w)) = f(\hat{f}(w)) = w$. But f is one-to-one on D, thus

$\hat{f}(w) = g(w)$ on $N_{r_2}(w_0)$. Thus, $\hat{f}$ is analytic on $N_{r_2}(w_0)$ with $\hat{f}'(w) = \dfrac{1}{f'(\hat{f}(w))}$ for

$w \in N_{r_2}(w_0)$. This is valid for every w_0 in f(D); therefore, $\hat{f}$ is analytic on f(D) with

$\hat{f}'(w) = \dfrac{1}{f'(\hat{f}(w))}$ for $w \in f$(D).

Q.E.D.

There are stronger versions of the previous two theorems which eliminate the assumption that $f'(z) \neq 0$.

Open Mapping Theorem

Let D be an open subset of C and f:D $\rightarrow$ C be nonconstant and analytic. Then f is an open mapping.

Inverse Function Theorem

If f:D $\rightarrow$ C is analytic and one-to-one, then $f'(z) \neq 0$ for any $z \in$ D and f has an analytic inverse

$$\hat{f} : f(D) \rightarrow D \text{ such that}$$

$$\hat{f}'(w) = \frac{1}{f'(\hat{f}(w))}.$$

The proofs of these results are beyond the scope of this paper (5).

Corollary 4.2

If f is analytic on $N_r(z_0)$ with $f'(z_0) \neq 0$, then there exists $r_1 > 0$, $r_1 \leq r$, such that

(1) f is one-to-one on $N_{r_1}(z_0)$,

(2) $f'(z) \neq 0$ on $N_{r_1}(z_0)$ and

(3) f has an analytic inverse $\hat{f}$ on $N_{r_1}(z_0)$ such that

$$\hat{f}'(w) = \frac{1}{f'(\hat{f}(w))} \quad \text{for } w \in N_{r_1}(z_0).$$

Proof

By Theorem 4.3 there exists $r_1 > 0$ such that $f'(z) \neq 0$ and f is one-to-one on $N_{r_1}(z_0)$. Now apply the Inverse Function Theorem to f on $D = N_{r_1}(z_0)$.

Q.E.D.

Corollary 4.3

Let f be analytic on the domain D with $f'(z) \neq 0$ on D. Then for each $z_0 \in D$, there exists $r_1 > 0$ such that f has an analytic inverse $\hat{f}$ on $N_{r_1}(z_0)$ and

$$\hat{f}'(w) = \frac{1}{f'(\hat{f}(w))} \quad \text{on } N_{r_1}(z_0).$$

Proof

Since D is an open set there exists $r > 0$ such that $N_r(z_0) \subset D$. Hence, f is analytic on $N_r(z_0)$ with $f'(z_0) \neq 0$. Corollary 4.3 now follows from Corollary 4.2.

Q.E.D.

The following results illustrates an application of the theorems presented in this chapter.

Theorem 4.6

Let D be a domain in **C**, f and g:D $\to$ **C** be two analytic functions on D. Suppose there exists $z_0 \in$ D such that

(1) $f(z_0) = g(z_0)$,

(2) $f'(z_0) \neq 0,\ g'(z_0) \neq 0$ and $\dfrac{f'(z_0)}{g'(z_0)}$ is a positive real number,

(3) for any $\delta > 0$ there exists $r > 0$ with $r \leq \delta$ such that $f(N_r(z_0)) = g(N_r(z_0))$, then $f(z) = g(z)$ for every $z \in$ D.

Proof

According to the first corollary following the Inverse Function Theorem applied to both f and g, there exists $\delta > 0$ with $N_\delta(z_0) \subset D$ such that $f'(z) \neq 0$, $g'(z) \neq 0$ on $N_\delta(z_0)$. and f and g are both one-to-one on $N_\delta(z_0)$. Then there exists $r > 0$ with $r \leq \delta$ such that $f(N_r(z_0)) = g(N_r(z_0))$. Since $N_r(z_0) \subset N_\delta(z_0)$, f and g are one-to-one on $N_r(z_0)$ and $f'(z) \neq 0$, $g'(z) \neq 0$ on $N_r(z_0)$.

If $D_1 = f(N_r(z_0)) = g(N_r(z_0))$, then f and g have analytic inverses $\hat{f}, \hat{g}: D_1 \to N_r(z_0)$ by the Inverse Function Theorem. Moreover ,

$$\hat{f}'(w) = \frac{1}{f'(\hat{f}(w))} \quad \text{and} \quad \hat{g}'(w) = \frac{1}{g'(\hat{g}(w))} .$$

Next consider the function T: $N_1(0) \to C$ defined by T(z)= rz + z_0. Then T is analytic such that $T'(z) = r > 0$,T is a one-to-one function and T(N₁(0)) = $N_r(z_0)$. To see this last asseration, if $|z| < 1$, then $|T(z) - z_0| = |rz| = r|z| < r$. Hence, if $z \in N_1(0)$, then $T(z) \in N_r(z_0)$. Thus, $T(N_1(0)) \subset N_r(z_0)$. If $w \in N_r(z_0)$, then $|w - z_0| < r$, and if $z = \frac{1}{r}|w - z_0| < 1$ and $T(z) = rz + z_0 = r(\frac{1}{r}(w - z_0)) + z_0 = w$.

Thus if $w \in N_r(z_0)$, there exists $z \in N_1(0)$ such that T(z) = w. This proves $N_r(z_0) \subset T(N_1(0))$. Consequently, T: $N_1(0) \to N_r(z_0)$ is an analytic function with analytic inverse $\hat{T}: N_r(z_0) \to N_1(0)$ where $\hat{T}(w) = \frac{1}{r}(w - z_0)$. Furthermore, T(0) = z_0. Define functions f_1 and g_1 to be the compositions

f_1 and g_1, $f_1 = f \circ T$, $g_1 = g \circ T: N_1(0) \to N_r(z_0) \to D_1$.

Then f_1 and g_1 have analytic inverses

$\hat{f}_1 = (f \circ T)^{-1} = \hat{T} \circ \hat{f}$, $\hat{g}_1 = \hat{T} \circ \hat{g}: D_1 \to N_1(0)$.

Moreover, $f_1(0) = f(T(0)) = f(z_0) = g(z_0) = g_1(0)$,

$$f_1'(z) = f'(T(z))T'(z) = rf'(T(z)) \quad \text{and} \quad g_1'(z) = rg'(T(z)) .$$

Next, let $h = \hat{g}_1 \circ f_1: N_1(0) \to D_1 \to N_1(0)$. This is an analytic function is such that $|h(z)| < 1$ on $N_1(0)$ and $h(0) = \hat{g}_1(f_1(0)) = \hat{g}_1(g_1(0) = 0$.

An application of Schwarz's Lemma to h shows $|h(z)| \le |z|$ for $z \in N_1(0)$. The same argument can be used for $\hat{h} = \hat{f}_1 \circ g_1$, the inverse of h, to show

$\left|\hat{h}(w)\right| \le |w|$ for every $w \in N_1(0)$. If $z \in N_1(0)$ and w = h(z) , then

$|h(z)| \le |z|$ and $|z| = \left|\hat{h}(h(z))\right| = \left|\hat{h}(w)\right| \le |w| = |h|z||$.

Therefore, $|h(z)| = |z|$ for $z \in N_1(0)$. Again by Schwarz's Lemma there exists $c = e^{i\phi}$ such that with $|c| = 1$. h(z) = cz.

Therefore h ′(z) = c. Now apply the chain rule to $h = \hat{g}_1 \circ f_1$,

$$h'(0) = (\hat{g}_1 \circ f_1)'(0) = \hat{g}_1'(f_1(0))f_1'(0)$$

$$= \frac{1}{g_1'(\hat{g}_1(f_1(0)))} rf\,'(T(0))$$

$$= \frac{1}{rg'(T(0))} rf\,'(T(0))$$

$$= \frac{f\,'(z_0)}{g'(z_0)} > 0 .$$

Thus $h'(z) = c$ is a positive real number and since $|c| = 1$, then c = 1.

Consequently h(z) = z for every $z \in N_1(0)$. But

$$h(z) = (\hat{g}_1 \circ f_1)(z) = \hat{g}_1(f_1(z)) \text{ ; therefore ,}$$

$$g_1(z) = g_1(h(z)) = g_1(\hat{g}_1(f_1(z))) = f_1(z)$$

If $w \in N_r(z_0)$ and $z = \hat{T}(w) \in N_1(0)$, then $g_1(z) = f_1(z)$ implies

$$f(w) = f(T(\hat{T}(w))) = f_1(z) = g_1(z) = g(T(\hat{T}(w))) = g(w) .$$

Therefore $f = g$ on $N_r(z_0)$ and therefore $f = g$ on the whole domain D by the Proposition 3.2 which is listed in chapter 3.

Q.E.D.

Corollary 4.4

Suppose $f : \mathbf{C} \to \mathbf{C}$ is analytic such that $f(z_0) = z_0$ for some $z_0 \in \boldsymbol{C}$.

If (1) $f'(z_0)$ is a positive real number and

(2) for any $\delta > 0$ there exists $r > 0$ with $r \le \delta$ such that $f(N_r(z_0)) = N_r(z_0)$,

then $f(z) = z$ for every $z \in \mathbf{C}$.

This is proved by applying the last result to f and g = the identity function ($g(z) = z$).

CHAPTER 5

SUMMARY AND RECOMMENDATION

This thesis explains many of the properties of analytic functions of a complex variable, along with a presentation of examples and theorems that are necessary for the proofs of the three theorems. These three theorems form an important basis for the theoretical study of Julia sets and the Mandelbort set. This paper can serve as a starting point for such a study of Julia sets and the Mandelbort set in a future paper.

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