On the Solutions of Certain Difference Equations

Loretta I. Chue

ON THE SOLUTIONS OF CERTAIN DIFFERENCE EQUATIONS

THESIS

BY

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THESIS

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1999

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ON THE SOLUTIONS OF CERTAIN DIFFERENCE EQUATIONS

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The objective of this paper is to investigate the behavior of solutions to certain types of difference equations. Two types of equations will be studied. They are:

(i) $x_{n+1} = f(x_n)$ and

(ii) $x_{n+1} = g(x_{n-1}, x_n)$

where f and g are certain functions.

Properties to be investigated include boundedness, oscillation, periodicity and asymptotic behavior.

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CHAPTER 1

INTRODUCTION

The discrete analog to an ordinary differential equation is a difference equation. Ordinary difference equations are versatile tools of analysis. These equations are defined in terms of a single dynamic variable (that is, a single function of time) and, therefore, represent only a special case of more general dynamical models. They are excellent representations of many dynamic situations, and their associated theory is rich enough to provide substance to one's understanding. For example, scientists and mathematicians use observed or experimentally determined parameters to set up mathematical models based on differential or difference equations to determine the non-random, chaotic behavior of certain reactions.

An ordinary difference equation is a relation of the form:

$$y_{k+n} = F(k, y_{k+n-1}, y_{k+n-2}, \ldots, y_k) \tag{1a}$$

where F is a well-defined function of its arguments. Also, we denote the general member of the sequence by y_k and use the notation $\{y_k\}$ to represent the sequence $y_0, y_1, y_2, \ldots$ Thus, given appropriate starting values, all the remaining members of the sequence can be generated. In fact, from the above equation it can be seen that if n successive values of y_k are specified, then the sequence $\{y_k\}$ is uniquely defined. These specified values are called initial conditions. Therefore, part of the specification of a difference equation

is the set of integers k for which it is to hold. In general, this set of integers must be a sequence of successive values, of either a finite or infinite duration such as k = 0, 1, 2, 3, …, n or k = 0, 1, 2, 3 …..

The order of a difference equation is the difference between the highest and lowest indices that appear in the equation. The equation (1a) is an n^{th} order difference equation if and only if the term y_k appears in the function F on the right-hand side. For example,

$$y_{k+n+r} = F(k+r, y_{k+n+r-1}, y_{k+n+r-2}, \ldots, y_{k+r})$$

A solution of a difference equation is a function $\phi(k)$ that reduces the equation to an identity. Alternatively, a solution can be viewed as a sequence of numbers. The two viewpoints of a solution are of course equivalent. A difference equation need not necessarily possess a solution, and even if it does exist, there is no assurance that it will be unique. The solution must be further specified by giving a set of initial conditions equal in number to the order of the equation.

Finally, ordinary difference equations are quite adequate for the study of many problems, and the associated theory provides a good background for more general multi-variable theory. In other words, both with respect to problem formulation and theoretical development, difference equations of a single variable provide an important first step in developing techniques for the mathematical analysis of dynamic phenomena. In general, difference equations are expected to occur whenever the system under study depends on one or more variables that can only assume a discrete set of possible values.

CHAPTER 2

AN INVESTIGATION OF THE EQUATION

$x_{n+1} = f(x_n)$

To begin our study consider the equation:

$$x_{n+1} = ax_n,\ a \neq 0 \qquad (2a)$$

From equation (2.a) it is clear that if x_0 is an initial value, then:

$$x_1 = a\,x_0$$

$$x_2 = a\,x_1 = a(a\,x_0) = a^2\,x_0$$

$$x_3 = a\,x_2 = a(a^2\,x_0) = a^3\,x_0$$

$$\vdots$$

$$x_n = a\,x_{n-1} \ldots = a^n\,x_0$$

So the general solution of equation (2a) is given by the equation

$$x_n = a^n\,x_0 \qquad (2b)$$

Clearly if $x_0 = 0$, then x_n is the trivial solution, that is, $x_n = 0$ for every n. Assume that $x_0 \neq 0$ for the remainder of the chapter. It is worth mentioning that $x_n = f^n(x_0)$ where f^n denotes the n^{th} iterate of the function $f(x) = ax$. Thus, solutions of first order difference equations are the orbits of some x_0 in the domain of f.

Some properties mentioned before will be investigated and relevant terms will be defined.

Definition: The statement that the sequence $\{x_n\} \to \infty$ means that for each positive number m there exists N such that for all $n>N$, $x_n>m$

Definition: A sequence $\{x_n\}$ is said to converge to a real number L provided for each $\varepsilon > 0$ there exists a positive integer N such that $|x_n - L| < \varepsilon$ whenever $n > N$.

Lemma 2.1: If $a > 1$ then $a^n \to \infty$ as $n \to \infty$

Proof: Let $m > 0$ be given and assume $a > 1$

Then $a^n < a^{n+1}$

Let N be a positive integer such that

$$N > \frac{\log m}{\log a}$$

So, if $n > N$ then $n > \frac{\log m}{\log a}$

Which implies $a^n > m$ for all $n > N$

Therefore $\{a^n\} \to \infty$ as $n \to \infty$

Hence, the following theorem holds

Theorem 2.2: If $a > 1$ and $x_0 > 0$, then all solutions $\{x_n\}$ of equation (2a) satisfy the condition.

$$x_n \to \infty \text{ as } n \to \infty$$

<u>Lemma 2.3:</u> If $0 < a < 1$, then $\{a^n\} \to 0$ as $n \to \infty$

<u>Proof:</u> Let $\varepsilon>0$ be given

Since $0 < a < 1$, then $a^{n+1} < a^n$

Let N be a positive integer such that

$$N > \frac{\log \varepsilon}{\log a}$$

So, if $n > N$, then $n > \dfrac{\log \varepsilon}{\log a}$

$a^n < \varepsilon$ for all $n > N$

Since $0 < a^n$, then $|a^n| = a^n < \varepsilon$

Therefore $\{a^n\} \to 0$ as $n \to \infty$

<u>Theorem 2.4:</u> If $0 < a < 1$ and $x_0 > 0$, then all solutions x_n of equation (2.1) satisfy the condition $x_n \to 0$ as $n \to \infty$.

<u>Definition:</u> Let t be in the domain off. Then t is an equilibrium point if $f(t) = t$.

Note that for the function $f(x)=ax$, $f(0)=0$. Thus $x=0$ is an equilibrium point for the difference equation.

$$x_{n+1} = a\, x_n$$

Theorem 2.2 states that when $a > 1$, $x = 0$ is repelling but when $0 < a < 1$, $x = 0$ is an attracting equilibrium point, that is all solutions converge to 0 and so $x = 0$ is a global attractor.

Definition: p is a global attractor for the function f iff for each initial value x_0 the sequence $\{x_n\}$ converges to p where x_n is the n^{th} iterate of x_0 under f.

Moreover, Theorems 2.1 and 2.2 illustrate that solutions do not change signs and thus they are nonoscillatory. When $a < 0$ oscillatory behavior occurs.

Definition: A sequence $\{x_n\}$ is said to oscillate if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise, the sequence is called nonoscillatory.

Theorem 2.5: If $a < -1$ and $x_0 > 0$, then all solutions $\{x_n\}$ of equation (2a) are oscillatory and $\{|x_n|\} \to \infty$ as $n \to \infty$

Proof: From equation (2b) the general solution of equation (2a) is

$x_n = a^n x_0$

Since $a < 0$, $x_n \cdot x_{n+1} = a^n x_0 \cdot a^{n+1} x_0$

$= a^{2n+1} x_0^2 < 0$ for each n.

Thus $\{x_n\}$ is oscillatory.

For the remaining part of the theorem note that $|x_n| = |a^n x_0| = |a|^n \cdot x_0$ and since $|a| > 1$ by lemma 2.3, $\{|x_n|\} \to \infty$ as $n \to \infty$

Theorem 2.6: If $-1 < a < 0$ and $x_0 > 0$, then all solutions $\{x_n\}$ of equation (2a) are oscillatory and $x_n \to 0$

When $|a| = 1$, periodic solutions are obtained.

<u>Theorem 2.7:</u> If a=1, then every point is an equilibrium point, that is, $x_{n+1} = x_n = x_0$ for every n. If a = -1, then every solution has period 2, that is $x_{n+2} = x_n$ for all n and the orbit is the sequence $\{ x_0, -x_0, x_0, -x_0, \ldots\}$.

Thus when $|a| = 1$ all solutions of equation (2a) are periodic with period 1 or period 2. Hence, we have completely described all solutions of equation (2a).

Next, consider a piecewise linear map known as the Baker's Map when restricted to the interval [0, 1] was considered.

The piecewise linear map is defined by:

$$B(x) = \begin{cases} 2x, & x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x \end{cases} \qquad (2c)$$

Thus the equation $x^{n+1} = B(x_n)$ will be investigated.

For $x_0 < 0$,

$$x_{n+1} = B(x_n) = 2x_n$$

So $x_1 = 2x_0$

$x_2 = 4x_0$

$x_3 = 8x_0$

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$x_n = 2^n x_0 \rightarrow -\infty$

Note: If $x_n > 1$, then $x_{n+1} = 2x_n - 1 > 1$

For $x_0 > 1$

$$x_{n+1} = 2x_n - 1$$

So $x_1 = 2x_0 - 1$

$x_2 = 2x_1 - 1$

$= 2(2x_0 - 1) - 1$

$= 4x_0 - 3 \quad = 2^2 x_0 - 3$

$x_3 = 2x_2 - 1$

$= 2(4x_0 - 3) - 1$

$= 8x_0 - 7 = 2^3 x_0 - 7$

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$x_n = 2^n x_0 - (2^n - 1)$

$= 2^n (x_0 - 1) + 1 \to \infty$

Note that equation 2c has two equilibrium points $x = 0$ and $x=1$. Let x_0 denote the initial value of a solution x_n.

<u>Theorem 2.8</u>: If $x_0 < 0$, then $x_n \to -\infty$ as $n \to \infty$.

If $x_0 > 1$, then $x_n \to \infty$ as $n \to \infty$. Hence when

$|x_0 - ½| > ½$, solutions are unbounded.

Now consider the case when $|x_0 - ½| \leq ½$, that is, $0 \leq x_n \leq 1$

Theorem 2.9: If $x_0 \in I = [0, 1]$, then $x_n \in I = [0, 1]$ for all n.

Thus for $x_0 = [0, 1]$, solutions are bounded.

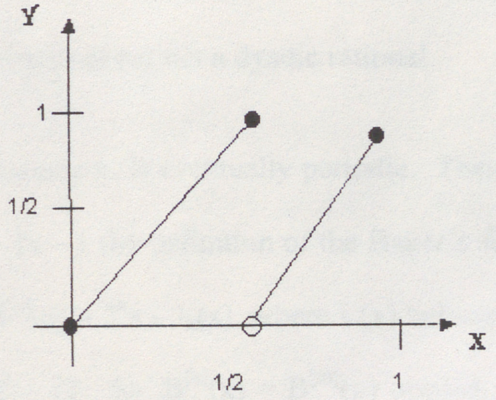
That is: $0 \leq x_n \leq 1$

Definition: A number x_0 in [0, 1] is called a dyadic rational if it has the form $k/2^m$ for some non-negative integers k and m.

Definition: Let x_0 be in the domain of f. Then x_0 is an eventually fixed point of f if there is a positive integer n such that $f^{[n]}(x)$ is an equilibrium point off.

Theorem 2.10: If $x_0 \in I = [0, 1]$ and x_0 is a dyadic rational then x_n is eventually fixed.

Graph of B(x)



Proof: Assume x_0 is a dyadic rational in [0, 1], then $x_0 = k/2^m$ for some nonnegative integers k and m. If $k = 0$ or $k = 2^m$,

then x_0 is an equilibrium point. Suppose $0 < k < 2^m$ · So, $B^{[m]}(x_0) = B^{[m]}(k/2^m) = 2^m(k/2^m) - i_m = k - i_m$. This implies $0 \leq k - i_m \leq 1$. Thus $k - i_m = 0$ or $k - i_m = 1$. So, x_n is eventually fixed. Conversely, assume x_n is eventually fixed, then for some n, $B^{[n]}(x_o) = 0$ or 1. Thus, $2^n x_o - k = 0$ or $2^n x_o - k = 1$. Solving for x_o, $x_o = k/2^n$ or $(k + 1)/2^n$, which is a dyadic rational. Hence, x_a is eventually fixed.

<u>Definition</u>: Let f be a function and let x_o be in the domain of f. Then x_o is a <u>periodic point</u> if there exists an n such that $f^{[n]}(x_o) = x_o$.

<u>Definition</u>: A number x_o is an <u>eventually periodic point</u> of a function f if there is a positive integer n such that $f^{[n]}(x)$ is periodic.

<u>Theorem 2.11</u>: x_n is eventually periodic if and only if $x_0 \in I = [0, 1]$ and x_0 is rational but not a dyadic rational.

<u>Proof</u>: Assume x_n is eventually periodic. Then, $B(x) = 2x$ or $B(x) = 2x - 1$ (by definition of the Baker's function). In general $B^{[n]}(x) = 2^n x - i_n(x)$, where $i_n(x)$ belong to the set $\{0, 1, \ldots, 2^n - 1\}$. So, $B^{[k]}(x) = B^{[m]}(x)$ implies $2^k x - i_k(x) = 2^m x - i_m(x)$. This implies $x = (i_k(x) - i_m(x))/(2^k - 2^m)$ which is

rational. Hence, if x_n is eventually periodic then $x_0 = x$ is rational. Conversely, suppose x_0 is rational, then x_n is eventually periodic. So, assume $x_0 = k/p$, where k and p are integers. Then, the definition of $B(k/p)$ implies that $B(k/p) = 2k/p$ or $(2k/p) - 1 = (2k - p)/p$. So, in either case n/p, where n is an integer, is obtained. Since there are only a finite collection of numbers of the form n/p between (0, 1), this implies that x_n is eventually periodic.

<u>Definition:</u> Let $f: [0, 1] \to [0, 1]$. The n^{th} iterate of f is denoted $f^{[n]}$. More precisely, $f^{[0]}(x) = x$, $f^{[1]}(x) = f(x)$, $f^{[2]}(x) = f(f(x))$, …, $f^{[n+1]}(x) = f(f^{[n]}(x))$.

<u>Definition:</u> The n^{th} iterate of the Baker's function $B^{[n]}$ is defined as follows:

$$B^{[n]}(x) = \begin{cases} 2^n x, \text{ for } 0 \le x \le ½^n \\ 2^n x - 1, \text{ for } ½^n < x \le 2/2^n \\ 2^n x - 2, \text{ for } 2/2^n < x \le 3/2^n \\ \cdot \\ \cdot \\ 2^n x - (2^n - 2), \text{ for } \frac{2^n - 2}{2^n} < x \le \frac{2^n - 1}{2^n} \\ 2^n x - (2^n - 1), \text{ for } \frac{2^n - 1}{2^n} < x \le 1 \end{cases}$$

<u>Theorem 2.12</u>: If $x_0 \in I = [0, 1]$ and x_0 is a irrational then the range of $\{x_n\}$ is infinite.

<u>Proof:</u> Assume x_0 is irrational

Then $0 < x_0 < 1$

Note that $x_1 = B(x_0) = \begin{cases} 2x_0, \text{ if } x_0 < ½ \\ 2x_0 - 1, \text{ if } x_0 > ½ \end{cases}$

So x_1 is irrational because of the properties of rational numbers.

Similarly, $x_n = B(x_{n-1})$ is irrational.

To show $B^i(x_0) \neq B^j(x_0)$

Assume $B^i(x_0) = B^j(x_0)$

So, $2^i x_0 - i_i(x_0) = 2^j x_0 - i_j(x_0)$

Then $2^i x_0 - 2^j x_0 = i_i(x_0) - i_j(x_0)$

$x_0 (2^i - 2^j) = i_i(x_0) - i_j(x_0)$

$x_0 = \dfrac{i_i(x_0) - i_j(x_0)}{2^i - 2^j}$ which implies that

x_0 is rational and, therefore, is a contradiction. Hence, the range of x_n is infinite.

CHAPTER 3

SOME PROPERTIES OF THE EQUATION

$$x_{n+1} = g(x_n, x_{n-1})$$

where $g(x, y) = \dfrac{ax + b}{(cs + d)y}$, a, b, c and d are nonnegative constants.

Some of the properties of the function above in conjunction with some theorems and examples will now be considered.

1. Periodic Solutions and Changing of Variables.

Consider the equation

$$x_{n+1} = \frac{ax_n + b}{(cx_n + d)x_{n-1}} \tag{3a}$$

With equation (3a) associate the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of the coefficients

The following are some interesting theorems along with their proofs.

Theorem 3.1: When the determinant of the matrix M is zero, every nontrivial solution of equation (3a) is periodic with (minimal) period 4.

Proof: Assume det M = 0, that is ad = bc, $B = \frac{b}{d}$, and assume also without loss of generality that $B(ax_n + b) = cx_n + d$. Then equation (3a) reduces to

$$x_{n+1} = \frac{B}{x_{n-1}}, \quad n = 0, 1 \dots \quad (3b)$$

So
$$x_{n+3} = \frac{B}{x_{n+1}} = B \cdot \frac{x_{n-1}}{B} = x_{n-1}$$

Therefore, $x_{n+4} = x_n$ which proves that every nontrivial solution of equation (3a) is periodic with (minimal) period 4.

Example: $x_{n+1} = \frac{2}{x_{n-1}}$, Let $x_{-1} = 1$, $x_0 = 2$

Then
$$x_1 = \frac{2}{x_{-1}} = 2$$

$$x_2 = \frac{2}{x_0} = 1$$

$$x_3 = \frac{2}{x_1} = 1$$

$$x_4 = \frac{2}{x_2} = 2$$

So $\{x_n\} = (2, 1, 1, 2)$

Theorem 3.2: When the main diagonal of the matrix M is zero, every nontrivial solution of equation (3a) is periodic with period 3.

Proof: Assume the main diagonal of the matrix M is zero, that is

$a = 0 = d, \qquad b\,c \neq 0$

Then equation (3a) reduces to the equation

$$x_{n+1} = \frac{B}{x_n x_{n-1}}, \quad n = 0, 1 \ldots . \quad (3c)$$

with $B = \frac{b}{c}$

So $\quad x_{n+2} = \frac{B}{x_{n+1}\, x_n}$.

$$x_{n+3} = \frac{B}{x_{n+2}\,x_{n+1}} = B \cdot \frac{1}{\frac{B}{x_{n+1}\,x_n} \cdot x_{n+1}}$$

$$= B \cdot \frac{x_n}{B}$$

Therefore, $x_{n+3} = x_n$

Hence, every nontrivial solution of equation (3a) is periodic with period 3.

Example: $x_{n+1} = \dfrac{1}{x_n x_{n-1}}$, $x_{-1} = 2$, $x_0 = 1$

$$x_1 = \frac{1}{x_0\,x_{-1}} = \tfrac{1}{2}$$

$$x_2 = \frac{1}{x_1\,x_0} = \frac{1}{\tfrac{1}{2}} = 2$$

$$x_3 = \frac{1}{x_2\,x_1} = \frac{1}{1} = 1$$

$\therefore \{x_n\} \equiv (\tfrac{1}{2}, 2, 1)$

Theorem 3.3: When the paradiagonal of the matrix M is zero, every nontrivial solution of equation (3a) is periodic with (minimal) period 6.

Proof: Assume the paradiagonal of the matrix m is zero, that is,

$b = 0 = c$, $ad \neq 0$

Then the equation (3a) reduces to the equation.

$$x_{n+1} = \frac{Ax_n}{x_{n-1}}, \quad n = 0, 1 \ldots . \quad (3d)$$

with $A = \dfrac{a}{d}$

$$x_{n+6} = \frac{Ax_{n+5}}{x_{n+4}} = A\left(\frac{Ax_{n+4}}{x_{n+3}}\right) \cdot \frac{1}{x_{n+4}}$$

$$= \frac{A^2 x_{n+4}}{x_{n+3}} \cdot \frac{1}{x_{n+4}}$$

$$= \frac{A^2}{x_{n+3}} = A^2 \cdot \frac{x_{n+1}}{Ax_{n+2}}$$

$$x_{n+6} = \frac{A^2 x_{n+1}}{Ax_{n+2}}$$

$$= \frac{Ax_{n+1}}{x_{n+2}}$$

$$= Ax_{n+1} \cdot \frac{x_n}{Ax_{n+1}} = \frac{Ax_{n+1} \cdot x_n}{Ax_{n+1}} = x_n$$

Hence, $x_{n+6} = x_n$ which proves that every nontrivial solution of equation (3a) is periodic with (minimal) period 6.

<u>Example</u>: $x_{n+1} = \frac{2x_n}{x_{n-1}}$, $A = 2$, $x_{-1} = 2$ and $x_0 = 3$

$$x_1 = \frac{2x_0}{x_{-1}} = \frac{2 \cdot 3}{2} = \frac{6}{2} = 3$$

$$x_2 = \frac{2x_1}{x_0} = \frac{2 \cdot 3}{3} = \frac{6}{3} = 2$$

$$x_3 = \frac{2x_2}{x_1} = \frac{2 \cdot 2}{3} = \frac{4}{3} = 1^1/_3$$

$$x_4 = \frac{2x_3}{x_2} = \frac{2 \cdot 4/3}{2} = 1^1/_3$$

$$x_5 = \frac{2x_4}{x_3} = \frac{2 \cdot 4/3}{4/3} = \frac{8}{3} \cdot \frac{3}{4} = 2$$

$$x_6 = \frac{2x_5}{x_4} = \frac{2 \cdot 2}{4/3} = 4 \cdot 3/4 = 3$$

Hence $\{x_n\} \equiv (3, 2, 1^1/_3, 1^1/_3, 2, 3)$

<u>Theorem 3.4:</u> When either $b = 0$ and $d^2 = ac$ and $a^2 = bd$, every nontrivial solution of equation (3a) is periodic with period 5. Furthermore, the change of variables

$$x_n = \frac{a}{dy_n} \quad \text{reduces equation (3a)}$$

$$\text{to} \quad y_{n+1} = \frac{y_n + 1}{y_{n-1}}, \; n = 0, 1, \ldots \text{ (3e)}$$

<u>Proof:</u> $x_{n+1} = \dfrac{ax_n + b}{(cx_n = d)x_{n-1}}$

Assume $x_n = \dfrac{a}{dy_{n+1}}$ which implies $x_{n+1} = \dfrac{a}{dy_{n+1}}$

and $b = 0$ and $d^2 = ac$

$$\text{Then} \quad \frac{a}{dy_{n+1}} = \left[a \cdot \frac{a}{dy_n}\right] \times \frac{1}{\left[c \cdot \frac{a}{dy_n} + d\right]\frac{a}{dy_{n-1}}}$$

$$= \frac{a^2}{dy_n} \cdot \frac{1}{\frac{d^2a}{d^2y_ny_{n-1}} + \frac{a}{y_{n-1}}}$$

$$= \frac{a}{dy_n} \cdot \frac{1}{\frac{1}{y_ny_{n-1}} + \frac{1}{y_{n-1}}}$$

$$\frac{dy_{n+1}}{a} = \left[\frac{1}{y_ny_{n-1}} + \frac{1}{y_{n-1}}\right]\frac{dy_n}{a} = \left[\frac{1}{y_ny_{n-1}} + \frac{y_n}{y_ny_{n-1}}\right]\frac{dy_n}{a}$$

$$\frac{dy_{n+1}}{a} = \frac{dy_n}{ay_ny_{n-1}} + \frac{dy_ny_n}{ay_ny_{n-1}}$$

$$y_{n+1} = \frac{1}{y_{n-1}} + \frac{y_n}{y_{n-1}} = \frac{1+y_n}{y_{n-1}}$$

Hence $y_{n+1} = \dfrac{y_n + 1}{y_{n-1}}$

Now let $y_0 = a$ and $y_1 = b$

Then $y_2 = \dfrac{y_1 + 1}{y_0} = \dfrac{b+1}{a}$

$$y_3 = \frac{y_2 + 1}{y_1} = \left[\frac{b+1}{a} + 1\right] \cdot \frac{1}{b} = \frac{b+1+a}{ab}$$

$$y_4 = \frac{y_3 + 1}{y_2} = \left[\frac{b+1+a}{ab} + 1\right] \cdot \frac{a}{b+1} = \frac{b+1+a+ab}{b(b+1)}.$$

$$= \frac{(a+1)(b+1)}{b(b+1)}$$

$$= \frac{a+1}{b}$$

$$y_5 = \frac{y_4 + 1}{y_3} = \left[\frac{a+1}{b} + 1\right] \frac{ab}{b+1+a} = \frac{a^2+a+ab}{b+1+a}.$$

$$= \frac{a(a+1+b)}{a+1+b}$$

$$= a$$

So $y_5 = y_0 = a$

Thus by induction, it follows that $y_{n+5} = y_n$.

For every $n \in N$.

Hence, every nontrivial solution of equation (3a) is periodic with period 5.

<u>Example</u>: $$y_{n+1} = \frac{y_n + 1}{y_{n-1}}, \text{ Let } y_{-1} = 1,\ y_0 = 2$$

$$y_1 = \frac{y_0 + 1}{y_{-1}} = \frac{2+1}{1} = \frac{3}{1} = 3$$

$$y_2 = \frac{y_1 + 1}{y_0} = \frac{3+1}{2} = \frac{4}{2} = 2$$

$$y_3 = \frac{y_2 + 1}{y_1} = \frac{2+1}{3} = \frac{3}{3} = 1$$

$$y_4 = \frac{y_3 + 1}{y_2} = \frac{1+1}{1} = \frac{2}{2} = 1$$

$$y_5 = \frac{y_4 + 1}{y_3} = \frac{1+1}{1} = \frac{2}{1} = 2$$

So $\{y_n\} = (3, 2, 1, 1, 2)$

Theorem 3.5: When $a = 0$ and $b\,c\,d \neq 0$, the change of variables

$$x_n = \frac{dy_n}{c}$$

Reduces equation (3a) to the equation

$$y_{n+1} = \frac{B}{(y_n + 1)\,y_{n-1}}, \quad n = 0, 1, \ldots. \tag{3f}$$

with $B = \dfrac{bc^2}{d^3}$

Proof:

$$x_{n+1} = \frac{ax_n + b}{(cx_n + d)x_{n-1}}, \quad a = 0$$

Assume $x_n = \dfrac{dy_n}{c}$ which implies

$$x_{n+1} = \frac{dy_{n+1}}{c}$$

Then
$$\frac{dy_{n+1}}{c} = \frac{b}{\left[c \cdot \frac{dy_n}{c} + d\right]\frac{dy_{n-1}}{c}}$$

$$= b\,\frac{1}{\left[\frac{d^2y_ny_{n-1}}{c} + \frac{d^2y_{n-1}}{c}\right]}$$

$$= \frac{bc}{d^2} \cdot \frac{1}{y_ny_{n-1} + y_{n-1}}$$

$$\frac{dy_{n+1}}{c} = \frac{bc}{d^2} \cdot \frac{1}{y_ny_{n-1} + y_{n-1}}$$

So $y_{n+1} = \dfrac{bc^2}{d^3} \cdot \dfrac{1}{(y_n + 1)y_{n-1}}$ where $B = \dfrac{bc^2}{d^3}$

Hence $y_{n+1} = \dfrac{B}{(y_n + 1)y_{n-1}}$

<u>Theorem 3.6:</u> When $c = 0$ and $abd \neq 0$, the change of variables

$x_n = \dfrac{b}{ay_n}$ reduces equation (3a) to

$$y_{n+1} \quad \frac{Ay_n}{(1 + y_n)y_{n-1}}, \quad n = 0, 1\ldots. \tag{3g}$$

with $A = \dfrac{bd}{a^2}$

<u>Proof:</u> $x_n = \dfrac{b}{ay_n}$ which implies that $x_{n+1} = \dfrac{b}{ay_{n+1}}$

So
$$x_{n+1} = \frac{b}{ay_{n+1}} = \frac{\left(a \cdot \dfrac{b}{ay_n} + b\right)}{d}$$

$$= \frac{b + by_n}{dy_n} \cdot \frac{ay_{n-1}}{b}$$

$$= \frac{b}{d}\left(\frac{1 + y_n}{y_n}\right) \cdot \frac{ay_{n-1}}{b}$$

$$= \frac{a}{d}\left(\frac{1 + y_n}{y_n}\right) \cdot \frac{y_{n-1}}{1}$$

$$\therefore \frac{b}{ay_{n+1}} = \frac{a}{d}\left(\frac{1 + y_n}{y_n}\right) \cdot \frac{y_{n-1}}{1}$$

$$\therefore \frac{ay_{n+1}}{b} = \frac{d}{a} \cdot \frac{y_n}{1 + y_n} \cdot \frac{1}{y_{n-1}}$$

$$y_{n+1} = \frac{bd}{a^2}\left(\frac{y_n}{1 + y_n}\right) \cdot \frac{1}{y_{n-1}}$$

$$= \frac{Ay_n}{(1 + y_n)y_{n-1}} \text{ where } A = \frac{bd}{a^2}$$

Since $A = \frac{bd}{a^2}$

Then $$y_{n+1} = \frac{Ay_n}{(y_n+1)\, y_{n-1}}$$

<u>Theorem 3.7</u>: When a bcd $\neq 0$ the change of variables

$$x_n = \frac{dy_n}{c}$$

reduces equation (3a) to the equation

$$y_{n+1} = \frac{Ay_n + B}{(y_n+1)\, y_{n-1}}, \quad n = 0, 1, \ldots \tag{3h}$$

<u>Proof:</u> Let $A = \frac{ac}{d^2}$ and $B = \frac{bc^2}{d^3}$

Assume $x_n = \frac{dy_n}{c}$ which implies $x_{n+1} = \frac{dy_{n+1}}{c}$

Then $x_{n+1} = \frac{ax_n + b}{(cx_n + d)\, x_{n-1}}$ becomes

$$\frac{dy_{n+1}}{c} = \frac{\frac{ady_n}{c} + b}{\left[c \cdot \frac{dy_n}{c} + d\right] \frac{dy_{n-1}}{c}} = \frac{\frac{ady_n}{c} + b}{(dy_n + d)\, \frac{dy_{n-1}}{c}}$$

$$= \frac{ady_n + bc}{d^2\, (y_n + 1)\, y_{n-1}}$$

$$\frac{dy_{n=1}}{c} = \frac{ady_n + bc}{d^2 (y_n + 1) y_{n-1}}$$

So $\frac{dy_{n+1}}{c} = \frac{ady_n + bc}{d^2(y_n + 1)\, y_{n-1}}$

$$y_{n+1} = \frac{c(ady_n + bc)}{d^3(y_n + 1)\, y_{n-1}} = \frac{\frac{acd}{d^3} y_n + \frac{bc^2}{d^3}}{(y_n + 1)\, y_{n-1}}$$

$$= \frac{\frac{ac}{d^2} y_n + \frac{bc^2}{d^3}}{(y_n + 1)\, y_{n-1}}$$

Since $A = \frac{ac}{d^2}$ and $B = \frac{bc^2}{d^3}$

Then $y_{n+1} = \frac{Ay_n + B}{(y_n + 1)\, y_{n-1}}$

Remark 1: From the theorems above, it follows that the following statements are true.

(i) When $ad - bc = 0$, every nontrivial solution of equation (3a) is periodic with (minimal) period 4.

(ii) when $a = d - 0$, every nontrivial solution of equation (3a) is periodic with period 3.

(iii) when $b = 0 = c$, every nontrivial solution of equation (3a) is periodic with (minimal) period 6.

(iv) In all other cases, a change of variables of the form

$$x_n = \lambda y_n \text{ or} = x_n \ \frac{\lambda}{y_n}$$

reduces equation (3a) to an equation of the form of

equation (3h) with

$$A \cdot B \in [0, \infty) \text{ and } A + B > 0 \qquad (3i)$$

and with arbitrary positive initial conditions y_{-1} and y_0.

2. An Invariant for $x_{n+1} = \dfrac{ax_n + b}{(cx_n + d)\, x_{n-1}}$

In this section it is shown that equation (3a) possesses the following invariant.

$$(b + ax_{n-1} + ax_n + dx_{n-1}\, x_n) \left[c + \frac{d}{x_{n-1}} + \frac{d}{x_n} + \frac{a}{x_{n-1}\, x_n} \right] = \text{constant} \quad (3j)$$

That is, for every solution $\{x_n\}$ of equation (3a), there is the identity for

all $n = 0, 1, \ldots.$

$$(b + ax_{n-1} + ax_n + dx_{n-1}\, x_n) \left[c + \frac{d}{x_{n-1}} + \frac{d}{x_n} + \frac{a}{x_{n-1}\, x_n} \right]$$

$$= \quad (b + ax_{-1} + ax_0 + dx_{-1}\, x_0) \left[c + \frac{d}{x_{-1}} + \frac{d}{x_0} + \frac{a}{x_{-1}\, x_0} \right]$$

This invariant (or first integral) for equation (3a) is a powerful tool in the analysis of equation (3a). In particular relatively straightforward and simple proofs can be done to show that every solution of equation (3a) is bounded and persists.

Theorem 3.8: Every solution of equation (3a) satisfied equation (3j).

Proof: If the left-hand side of equation (3h) is denoted by I_n, then for all n = 0, 1, ….

$$I_{n+1} = (b + ax_n + ax_{n+1} + dx_n x_{n+1})\left[c + \frac{d}{x_n} + \frac{d}{x_{n+1}} + \frac{a}{x_n x_{n+1}}\right]$$

$$= \left[b + ax_n + \frac{a(ax_n + b)}{(cx_n + d)x_{n-1}} + \frac{dx_n(ax_n + b)}{(cx_n + d)x_{n-1}}\right]$$

$$\times \left[\frac{cx_n + d}{x_n} + \frac{d(cx_n + d)x_{n-1}}{(ax_n + b)} + \frac{a(cx_n + d)x_{n-1}}{x_n(ax_n + b)}\right]$$

$$= \frac{(ax_n + b)}{(cx_n + d)x_{n-1}}[(cx_n + d)x_{n-1} + a + dx_n]$$

$$\times \frac{(cx_n + d)}{(ax_n + b)x_n}[(ax_n + b) + dx_{n-1}x_n + ax_{n-1}]$$

$$= (b + ax_{n-1} + ax_n + dx_{n-1}x_n)\left[c + \frac{d}{x_{n-1}} + \frac{d}{x_n} + \frac{a}{x_{n-1}x_n}\right]$$

$$= I_n$$

and the proof is complete.

Remark 2: It is worth mentioning that while equation (3j) holds for equation (3a), it is only meaningful when $a + d > 0$. When $a = 0 = d$ as we have seen in theorem 3.2, equation (3a) reduces to the "singular case."

$$x_{n+1} = \frac{b/c}{x_{n-1}x_n}, \quad n = 0, 1, \ldots.$$

With $\frac{b}{c} > 0$, every nontrivial solution of which is periodic

with period 3, and so equation (3d) clearly has its own invariant.

$$x_{n-1} + x_n + \frac{b}{cx_{n-1}x_n},$$

Similar to the invariant for equation (3a), the invariant for

equation (3h)

$$y_{n+1} = \frac{Ay_n + B}{(y_n + 1)y_{n-1}}.$$

is expressed in the following theorem.

Theorem 3.9: When $A = \frac{ac}{d^2}$ and $B = \frac{bc^2}{d^3}$ the invariant for equation (3h) is

$$(B + Ay_{n-1} + Ay_n + y_{n-1}\, y_n)\left[\frac{ad}{A} + \frac{ad}{Ay_{n-1}} + \frac{ad}{Ay_n} + \frac{A}{y_{n-1} \quad yn}\right] \text{(3k)}$$

$=$ constant.

3. Boundedness and Persistence:

A sequence is bounded and persists if there exist positive constants p and Q such that $p \leq x_n \leq Q$ for $n = -1, 0, \ldots$. In this section the invariant (3j) is utilized to show that every solution of equation (3a) is bounded from above and also bounded away from zero by a positive constant.

Theorem 3.10: Every solution of equation (3a) is bounded and persists.

Proof: By remark 2, we need only show that every solution $\{y_n\}$ of equation (3h) is bounded and persists. From equation (3k) it can be seen that

$$\frac{B}{y_n} + y_n + \frac{A^2}{y_n} + \text{nonnegative terms is a positive}$$

number, constant for $n = 0, 1 \ldots$

Hence $\{y_n\}$ is bounded from above, and because either $B > 0$ or $A > 0$, it is clear that $\left\{\frac{1}{y_n}\right\}$ is also bounded from above. The proof is complete.

CHAPTER 4

CONCLUSION

In this thesis the behavior of the solutions to two types of difference equations, first order linear and second order non-linear equations were investigated. Considering the first type, it was observed that the initial values played a significant role in the outcome of the solutions and determined whether solutions were oscillatory or not. Several cases were considered and the results showed particular values of a in the function f(x) = ax which give rise to equilibrium points, periodic solutions and boundedness.

In the nonlinear case, periodic solutions were obtained by changing variables to make calculations less cumbersome. Furthermore, an invariant was discovered which was used to show that every solution of equation (3a) was bounded from above and also bounded away from zero by a positive constant.

Finally, difference equations is an exciting area of studies, enhanced by the computational capabilities of the computer. However, a vast amount of research remains to be done, in particular, with regard to the theory of nonlinear difference equations which is relatively unexplored.

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